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Relaxation of Static Problems

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Outline

- *Static problems*: Stresses are the main unknowns, *equilibrium* and *minimum complementary energy* are the main operative principle (method of forces)
- Focus on the static problem of limit analysis at collapse for given limit surface of the material
- Existence, attainment, *quasiconvexity*, relaxation?
- Fused silica glass as an example of a class of materials that lack quasiconvexity and undergo relaxation through stress patterning
- Explicit constructions and numerical experiments...

Kinematic and static problems

- *Hellinger-Reissner functional:*

$$J(u,\sigma) = \int_\Omega \Big(\sigma\cdot\varepsilon(u) - \chi(\sigma) - f\cdot u\Big)\,dx - \int_{\Gamma_D} \sigma\nu\cdot(u-g)\,d\mathcal{H}^{d-1} - \int_{\Gamma_N} h\cdot u\,d\mathcal{H}^{d-1}$$

- *Kinematic problem:* $\inf_u \sup_\sigma J(u,\sigma)$
- Convexity, $W = \chi^* \Rightarrow$ *Minimum potential energy*

$$\inf_u \Big\{ \int_\Omega \Big(W(\varepsilon(u)) - f\cdot u\Big)\,dx - \int_{\Gamma_N} h\cdot u\,d\mathcal{H}^{d-1},\ u = g \text{ on } \Gamma_D \Big\}$$

- Lower-semicontinuity ~ *(Morrey) quasiconvexity*
- Equilibrium satisfied in limit along min sequences

Kinematic and static problems

- *Static problem:* $\sup_{\sigma} \inf_{u} J(u, \sigma)$
- Equivalent formulation: u ~ Lagrange multipliers,

$$\sup_{\sigma} \left\{ \int_{\Gamma_D} \sigma\nu \cdot g \, d\mathcal{H}^{d-1} - \int_{\Omega} \chi(\sigma) \, dx \right\}$$

subject to: i) $\mathrm{div}\sigma + f = 0$, in Ω;

ii) $\sigma\nu = h$, on Γ_N.

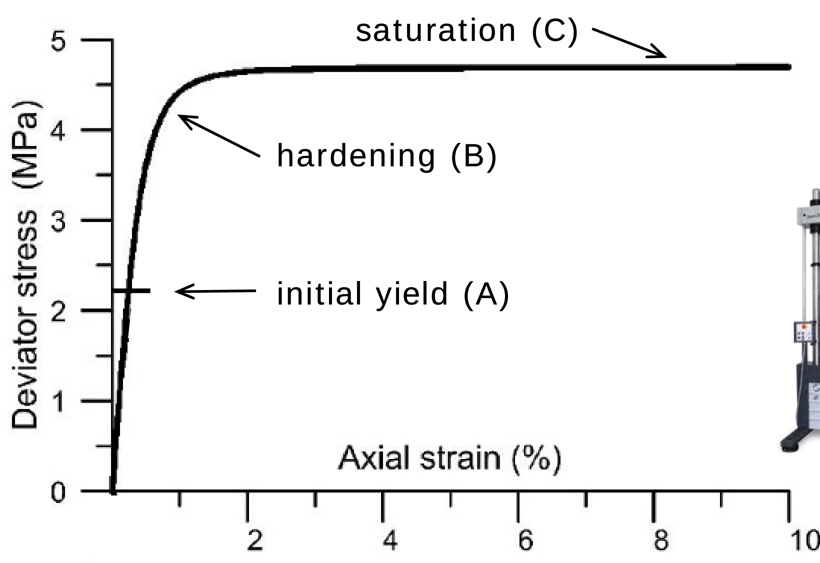
- Primary unknown: *Stress field*
- *Equilibrium* built into solution space *ab initio*
- Lower-semicontinuity ~ *sym-div quasiconvexity!*
- Complementary energy *not necessarily convex!*
- *Relaxation?* (in stress space!)

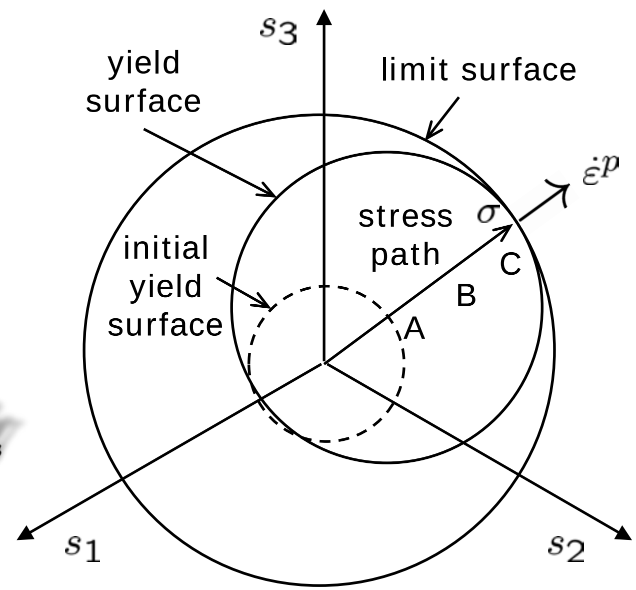
Application: Collapse of solids & structures

A steel arch bridge collapses in Yilan County, northeastern Taiwan, killing six people, on Oct. 1, 2019

Limit analysis at collapse

- *Stages of hardening:* Initial yield, power-law hardening and saturation





Typical stress-strain curve of elastic-plastic material showing transition from initial yield to saturation

Evolution of elastic domain from initial yield surface to *limiting surface* at saturation

Limit analysis at collapse

- *Definition of collapse:* Unconstrained incremental displacements (or velocities) at constant load

<u>Theorem.</u> *Assume elastic-plastic solid, convex elastic domain, rate-independent behavior. Then,*

$$\dot{f} = 0, \ \dot{g} = 0, \ \dot{h} = 0 \Rightarrow \dot{\sigma} = 0, \ \dot{\varepsilon}^e = 0, \ \dot{\varepsilon} = \dot{\varepsilon}^p.$$

- No incremental hardening (stress on *limit surface*)
- Behavior at collapse is *rigid-ideally plastic*
- Collapse signals the failure of *safe-load condition* and the *loss of coercivity* of rate functional
- *Static problem* for rate-independent rigid-plasticity

Limit analysis at collapse

- Recall, *general static problem*:

$$\sup_{\sigma} \left\{ \int_{\Gamma_D} \sigma\nu \cdot g \, d\mathcal{H}^{d-1} - \int_{\Omega} \chi(\sigma) \, dx \right\}$$

subject to: $\text{i)}\ \text{div}\sigma + f = 0, \ \text{in } \Omega;$

$$\text{ii)}\ \sigma\nu = h, \ \text{on } \Gamma_N.$$

- *Collapse*, limit surface *K*: $\chi(\sigma) = \begin{cases} 0, & \text{if } \sigma \in K, \\ +\infty, & \text{otherwise.} \end{cases}$
- *Static problem at collapse*:

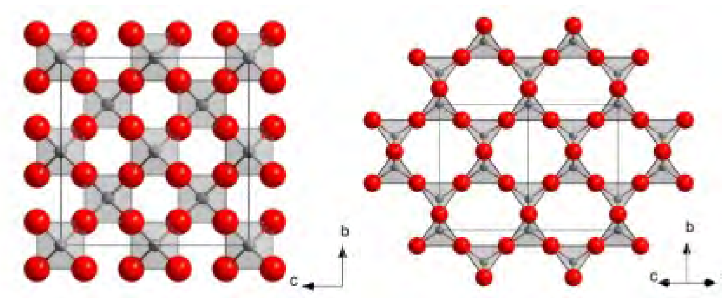
$$\sup_{\sigma} \left\{ \int_{\Gamma_D} \sigma\nu \cdot g \, d\mathcal{H}^{d-1} \ : \ \text{div}\, \sigma = 0, \ \sigma(x) \in K \right\}$$

- If *K* convex: Equivalent *kinematic problem*

Example: Fused silica glass

Starting structure: β-cristobalite

β-cristobalite: Polymorph characterized by **corner-bonded SiO_4 tetrahedra**

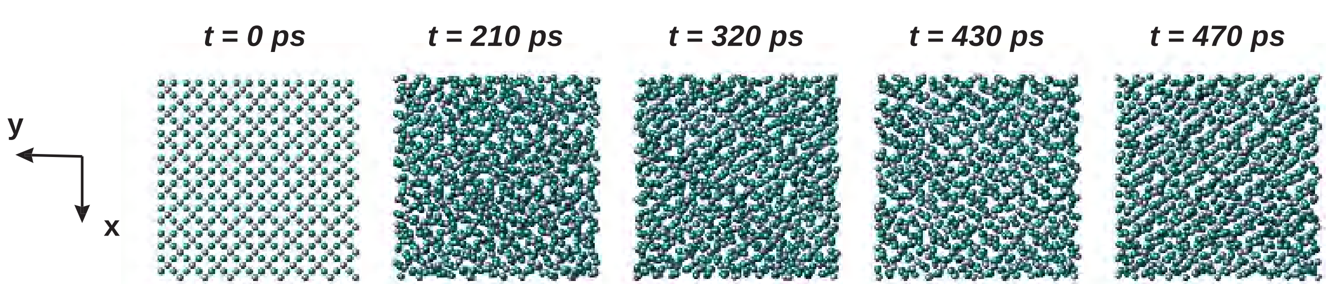


Ideal structure of β-cristobalite (adapted from https://en.wikipedia.org/wiki/Cristobalite)

Amorphous structure of fused silica: Obtained through the **fast quenching** of a melt

Steps taken during **quenching** process:

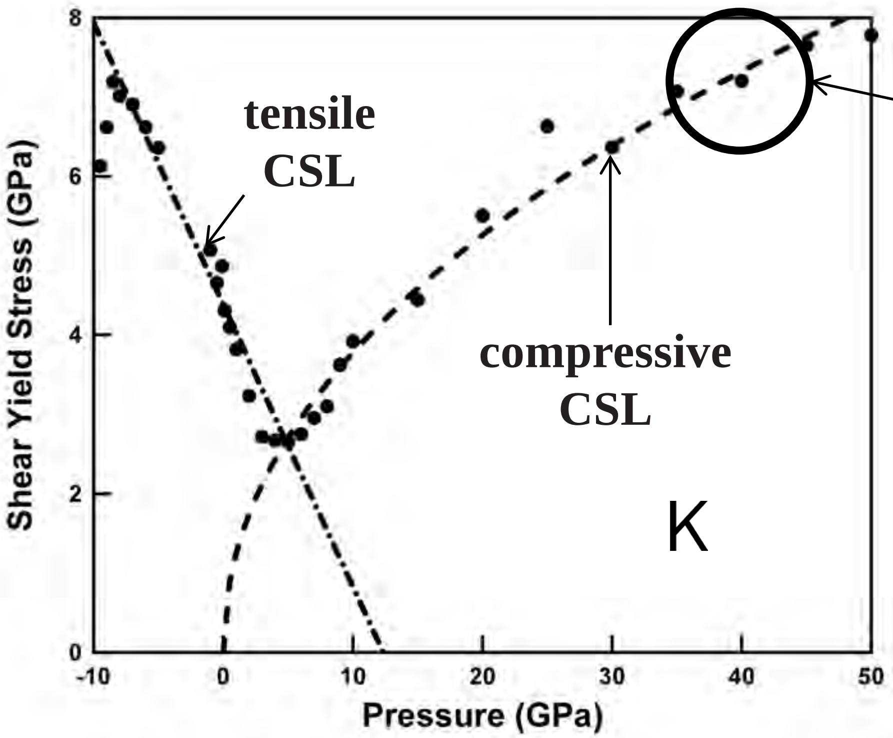
- Uniform temperature decrease from 5000 K to 300 K, decreasing the temperature with steps of 500 K
- Total cooling time: 470 ps

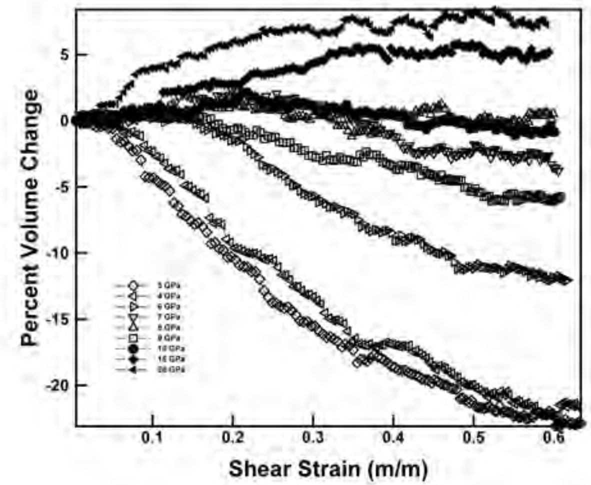


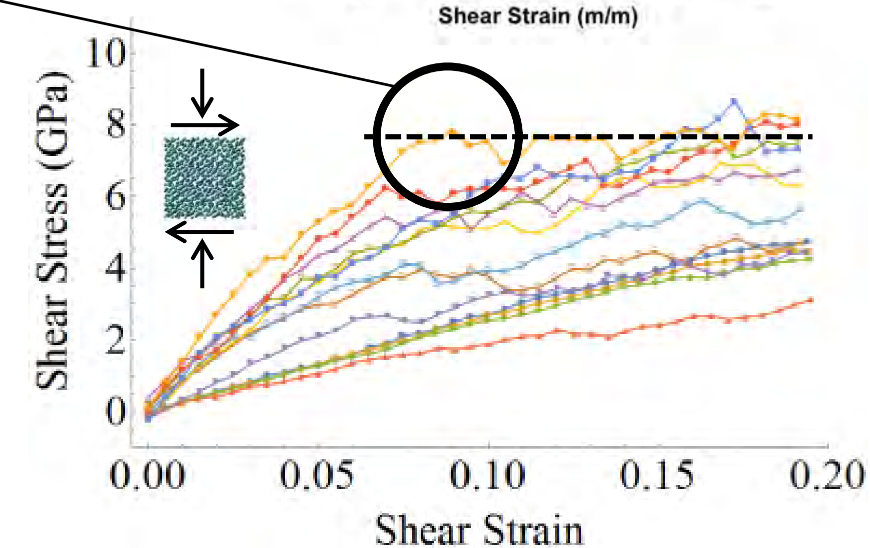
W. Schill and S. Heyden and S. Conti and M. Ortiz, *JMPS*, **113** (2018) 105-125.

Example: Fused silica glass

- *Pressure-dependent plasticity:* Critical states (yielding at constant volume) define Critical State Line (CSL)
- CSL bounds the limit domain *K*







Limit domain K strongly non convex!

Example: Fused silica glass

Effect of a Coordination Change on the Strength of Amorphous SiO_2

CHARLES MEADE AND RAYMOND JEANLOZ

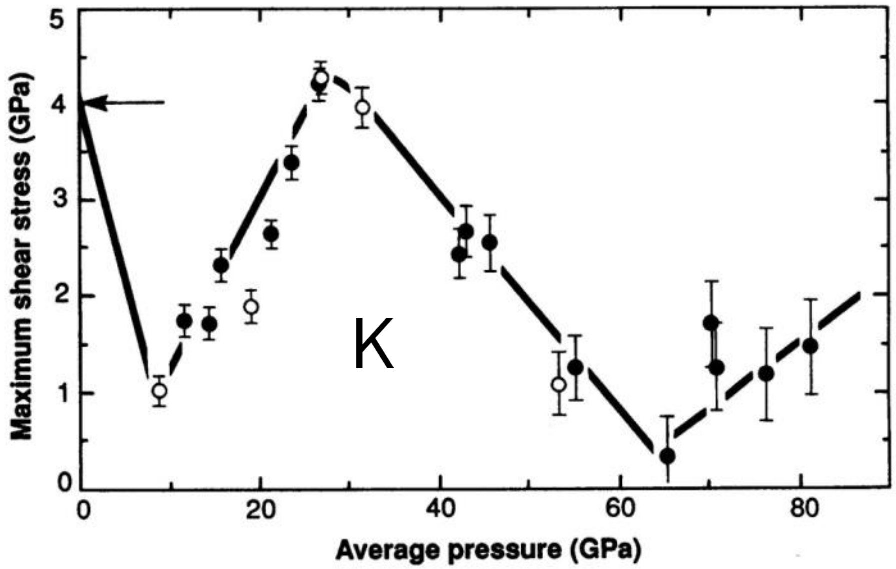


Fig. 1. Maximum shear stress in silica glass at room temperature and average pressures ($\bar{P}$) between 8.6 and 81 GPa. Each point corresponds to a separate sample, and the heavy line shows the general trend of the data. The shear stress is determined from Eq. 1, and it is a measure of the yield strength of the sample at high pressures. The error bars represent the combined uncertainties from the measurements of h and $\partial P/\partial r$. The open circles show the strength of samples that were initially compressed to 50 GPa, unloaded, and then recompressed. The arrow marks the zero pressure strength of silica glass (*19*).

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Anomalous shear strength of fused silica glass is well-documented in the geophysics literature

Symmetric div-quasiconvexity

- Recall, static problem at collapse:

$$\sup_{\sigma} \left\{ \int_{\Gamma_D} \sigma\nu \cdot g \, d\mathcal{H}^{d-1} \; : \; \mathrm{div}\, \sigma = 0, \;\; \sigma(x) \in K \right\}$$

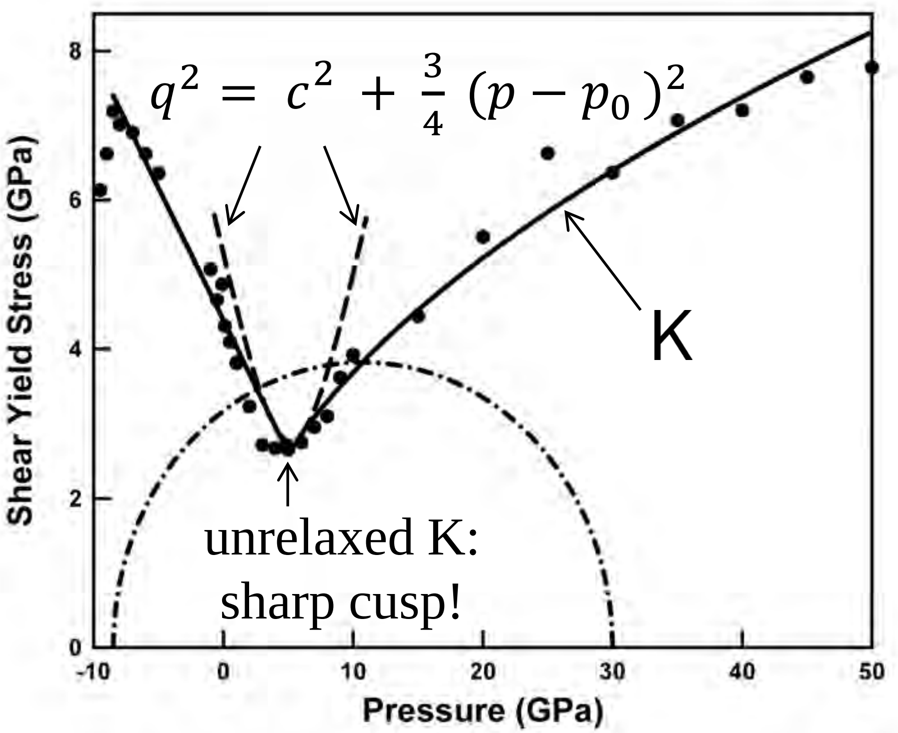
- If K is *strongly non-convex*, non-attainment.
- What is an appropriate notion of *quasiconvexiy*, *relaxation, envelops, constructions*?
- *Symmetric div-quasiconvexity* (sdqc), a special case of *$\mathcal{A}$-quasiconvexity* (Fonseca-Müller, 1999), implies *existence of solutions* of the static problem
- Relaxed problem defined by *sdqc envelop* K^{sdqc}
- *Exact constructions* for isotropic domains...

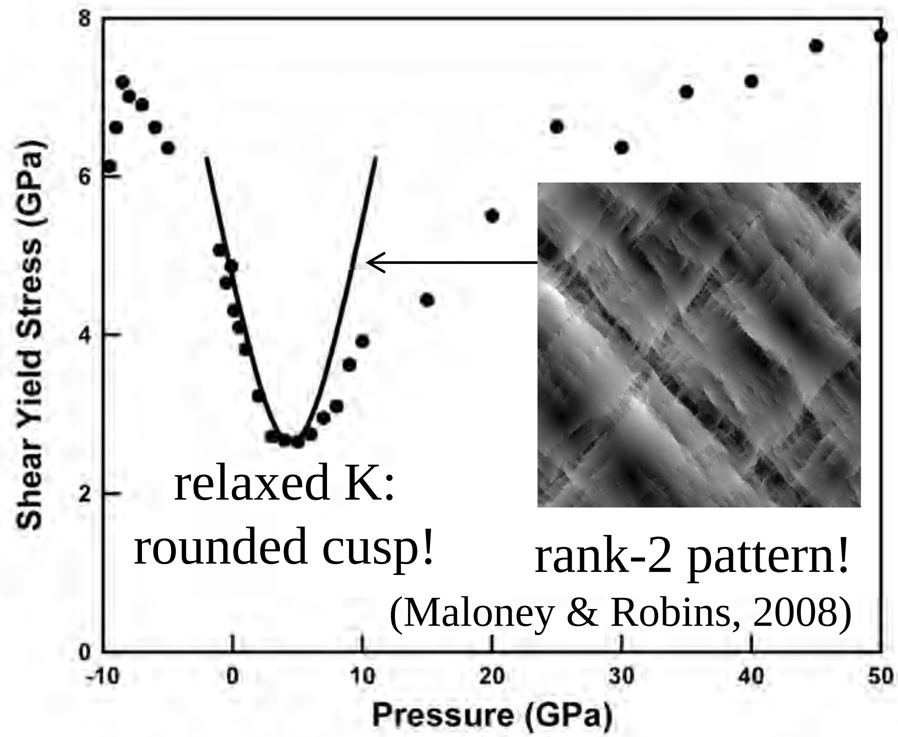
S. Conti, S. Muller and M. Ortiz, *ARMA*, **235** (2020) 841-880.

Symmetric div-quasiconvexity

Div-quasiconvex envelop of glass elastic domain:

- **Theorem** (Tartar'85). *The function* $f(\sigma) = 2|\sigma|^2 - \mathrm{tr}(\sigma)^2$ *is div-quasiconvex.*
- **Theorem**. *The set* $\{\sigma : q^2 \leq c^2 + \frac{3}{4}(p - p_0)^2\}$ *is sym div-quasiconvex.*
- **Theorem**. *The sym div-quasiconvex envelop* K^{sdqc} *of* $K(Si_2O)$ *is:*





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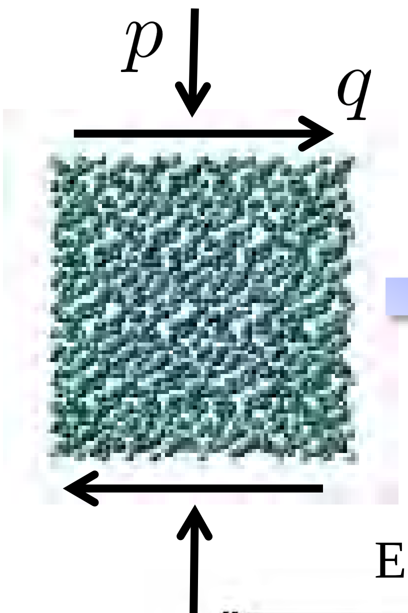
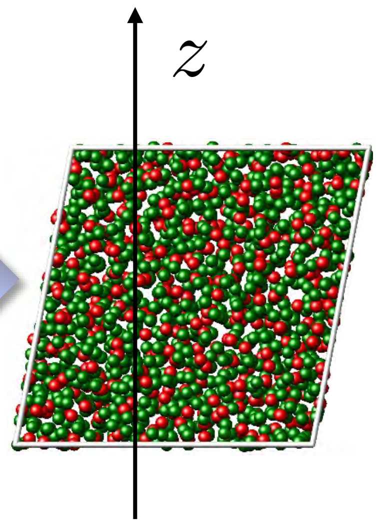
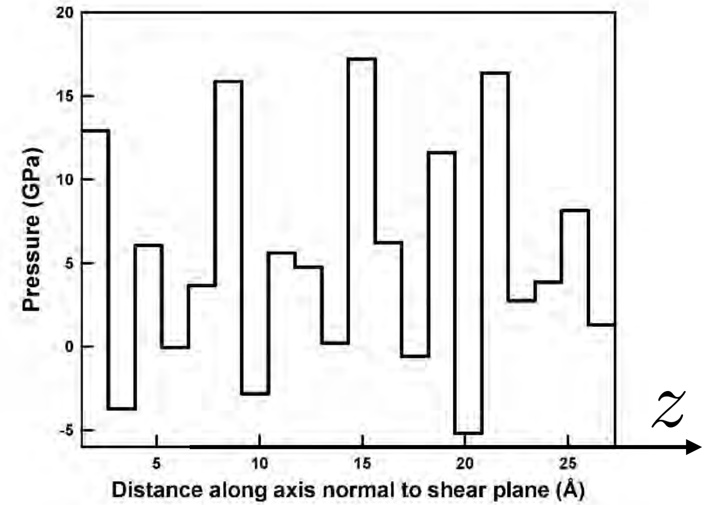
Symmetric div-quasiconvexity

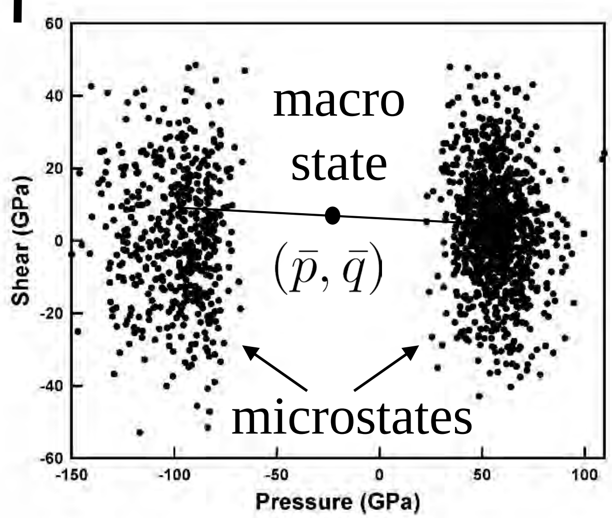


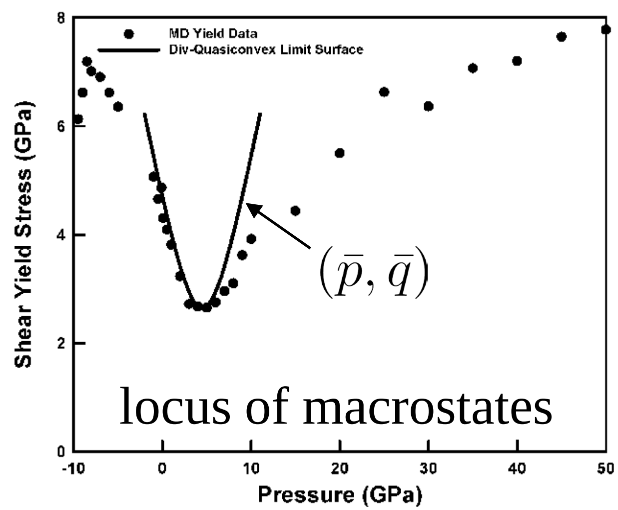


Equilibrium stress field



Pressure oscillations





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Concluding Remarks

- Static problems lacking attainment have been studied relatively less than kinematic problems
- Symmetric div-quasiconvexity provides a practical and compelling illustration of the abstract concept of $\mathcal{A}$-quasiconvexity of Fonseca and Müller
- Static problems set forth a non-standard type of relaxation achieved by means of equilibrated fine-oscillations in stress space
- The isotropic case is amenable to explicit envelop constructions that characterize rigorously and efficiently the macroscopic effective behavior of large classes of materials...

Can we prevent collapse?

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SIAM/MS 2021