Minimum principles for characterizing the trajectories and microstructural evolution of dissipative systems

M. Ortiz
California Institute of Technology
In collaboration with: **S. Conti, C. Larsen, A. Mielke, C. Richardson**

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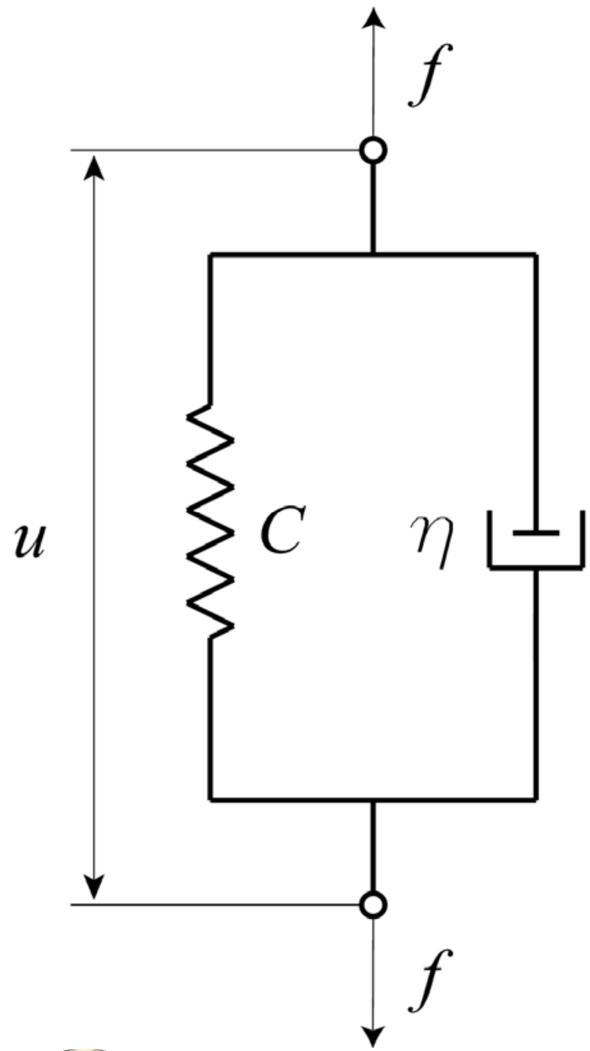
Systems with evolving microstructure

- The behavior of the systems of interest is governed by both energy and kinetics, e. g., through an **equation of evolution** of the form

$$\partial \Psi(\dot{u}) + DE(t,u) = 0, \begin{cases} \Psi \equiv \textit{dissipation potential} \\ E \equiv \textit{energy} \end{cases}$$

- However: Energies of interest often lack differentiability and lower-semicontinuity
- Meaning of equation of evolution, ‘solutions’?
- Effective macroscopic kinetics?
- Wanted: a theoretical framework that extends CoV to dissipative problems...

Classical rate variational problems



- Potential energy:

$$E(t,u) = \frac{C}{2}u^2 - f(t)u$$

- Dissipation potential: $\Psi(v) = \frac{\eta}{2}v^2$
- Rate potential:

$$G(t,u,v) \equiv \Psi(v) + DE(t,u)\,v$$

- Rate problem: Given t, u,

$$\min_v G(t,u,v)$$

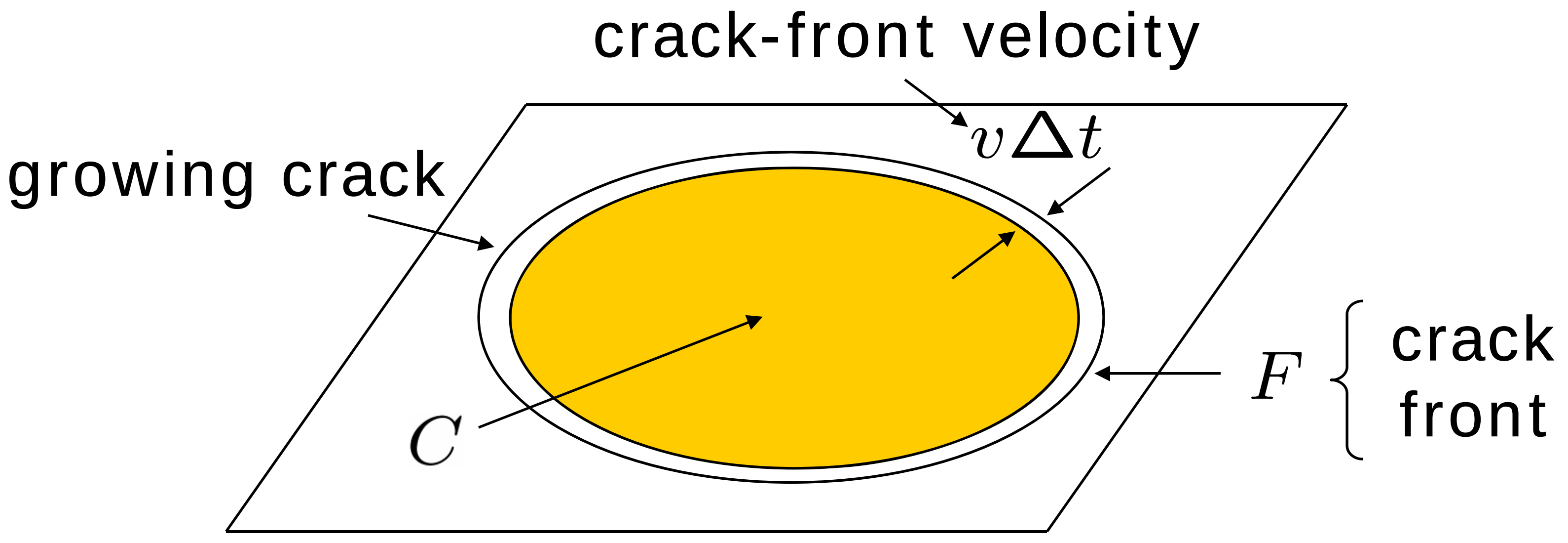
- Euler-Lagrange equations:

$$\partial\Psi(v) + DE(t,u) = 0$$

Energy-dissipation functionals

- Energy-dissipation functional: For $\epsilon > 0$,
$$F_\epsilon(u) = \int_0^T \mathrm{e}^{-t/\epsilon} G(t, u, \dot{u})\, dt$$
- Minimum principle: $\mathbb{Y} = \{u : [0, T] \to Y\}$,
$$\boxed{\inf_{u \in \mathbb{Y}} F_\epsilon(u)}$$
- Euler-Lagrange eqs., $G(u, \dot{u}) = \Psi(\dot{u}) + DE(u)\dot{u}$,
$$\underline{-\epsilon D^2\Psi(\dot{u})\ddot{u}} + D\Psi(\dot{u}) + DE(t, u) = 0$$
- Relaxation: $sc^- F_\epsilon(u) \stackrel{?}{=} \int_0^T \mathrm{e}^{-t/\epsilon} \underline{\bar{G}_\epsilon(t, u, \dot{u})\, dt}$
- Causal limit: $\bar{G}_\epsilon \to \bar{G}$ as $\epsilon \to 0$?

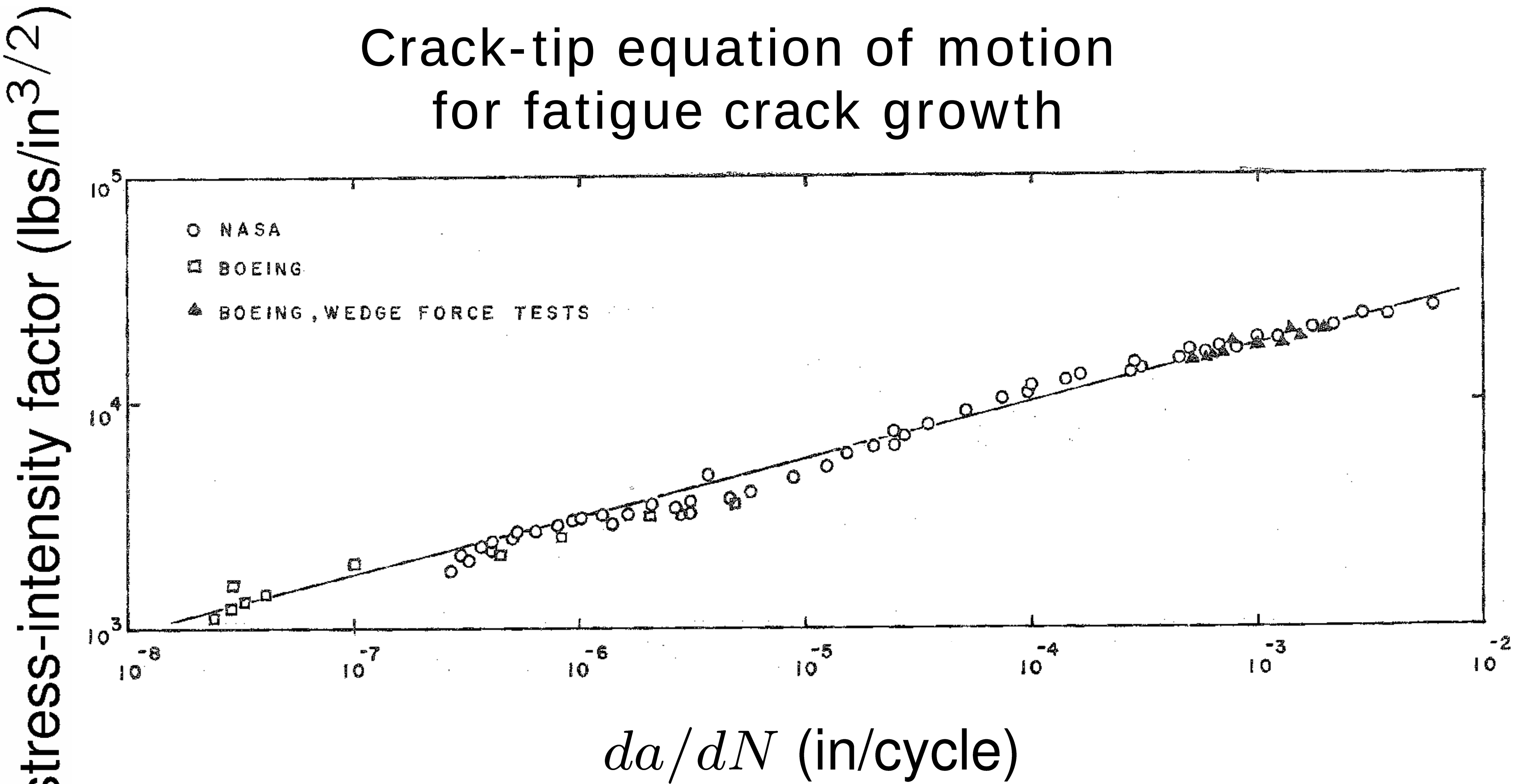
LEFM energy-dissipation functionals



- Energy: $E(u) = \int_\Omega W(\nabla u)\, dx +$ forcing terms
- Dissipation: $\Psi(v) = \int_F \psi(v)\, d\mathcal{H}^{n-2}$
- Crack front F, velocity v, defined distributionally

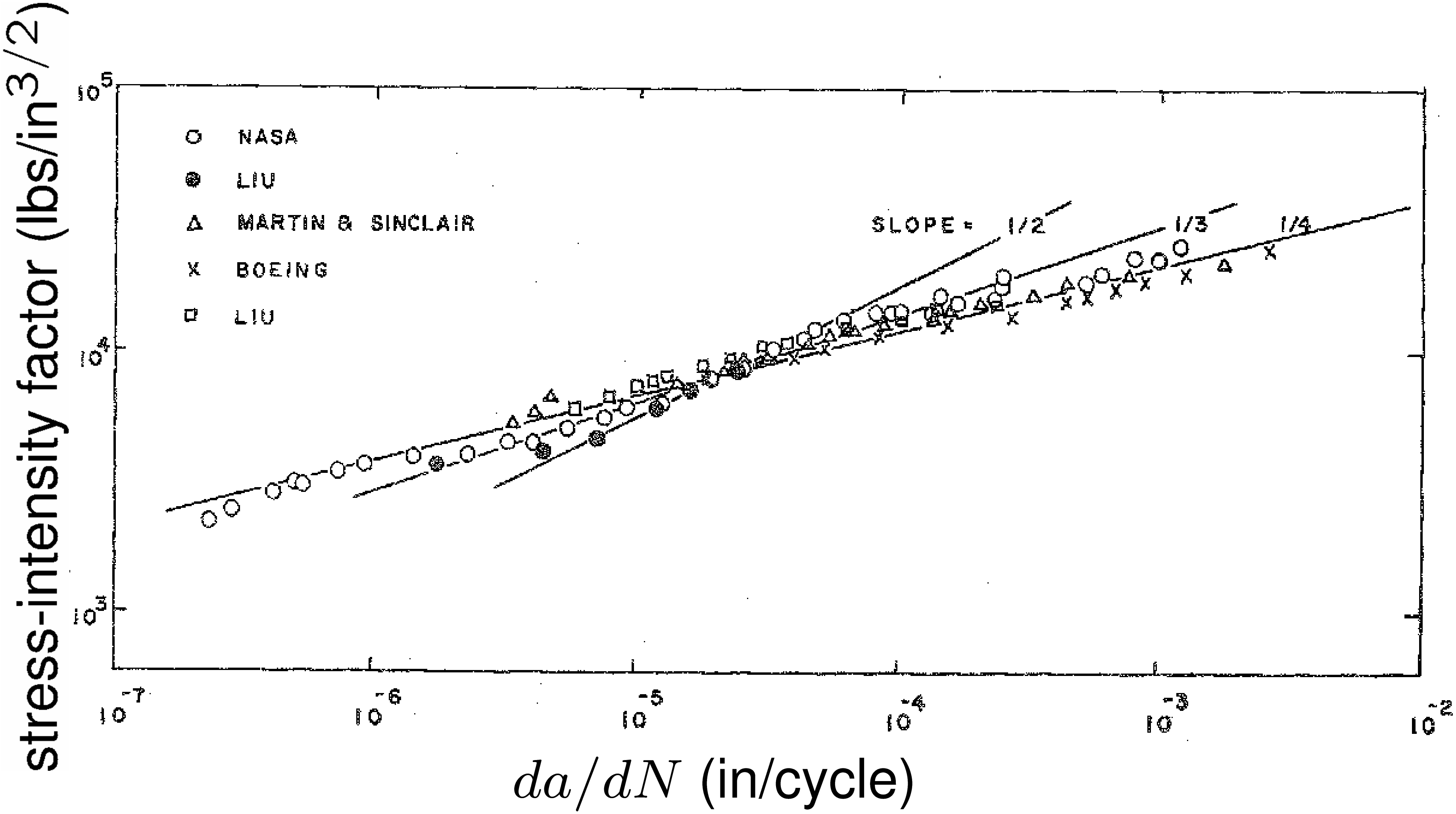
The rate problem of LEFM

Crack-tip equation of motion
for fatigue crack growth



Crack-growth data for 2024-T3 aluminum alloy
(P. Paris and F. Erdogan, ASME Trans (1963)

The rate problem of LEFM



Crack-growth data for 2024-T3 aluminum alloy
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LEFM energy-dissipation functionals

- Dissipation: $\Psi = \int_{F(t)} (\alpha + v^p)\, d\mathcal{H}^{n-2}$

nucleation energy

rate-dependent crack-tip equation of motion

- Energy-dissipation functional: $F_\epsilon(u) :=$

$$\int_0^T \mathrm{e}^{-t/\epsilon} \left\{ \int_{F(t)} (\alpha + v^p) d\mathcal{H}^{n-2} + \frac{1}{\epsilon} \int_\Omega W(\nabla u)\, dx \right\} dt$$

- Trajectories: $\mathbb{Y} \sim \{u(t) \in SBV_p(\Omega), \text{crack increasing}\}$
- Variational problem: $\boxed{\inf_{u \in \mathbb{Y}} F_\epsilon(u)}$

Michael Ortiz
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LEFM energy-dissipation functionals

Theorem (C. Larsen, MO, C.L. Richardson) *The lower semicontinuous envelop of F_ϵ in $\mathbb{Y}$ is:*

$$sc^- F_\epsilon(u) =$$

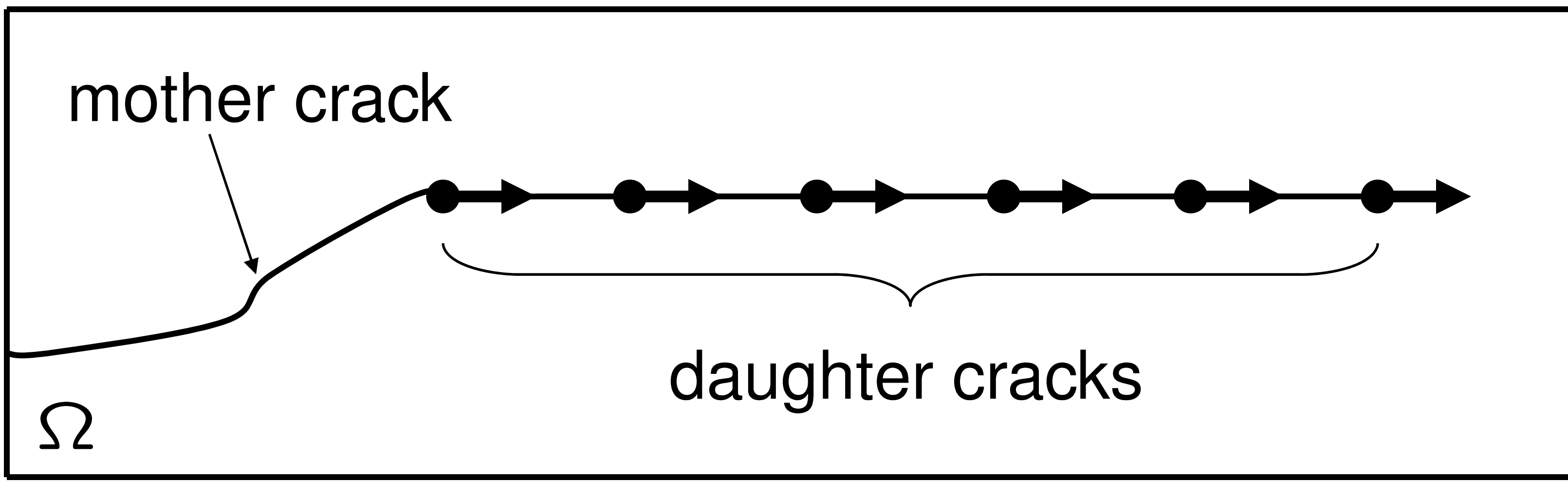
$$\int_0^T e^{-t/\epsilon} \left\{ \frac{1}{\epsilon} \int_\Omega W(\nabla u) dx + \gamma \int_{F(t)} v d\mathcal{H}^{n-2} \right\} dt$$

where: $\gamma = p \left(\frac{\alpha}{p-1} \right)^{\frac{p-1}{p}}$

- Relaxed energy-dissipation functional is **rate-independent**! (Griffith brittle fracture)

LEFM energy-dissipation functionals

Sketch of proof: Mother-daughter mechanism:



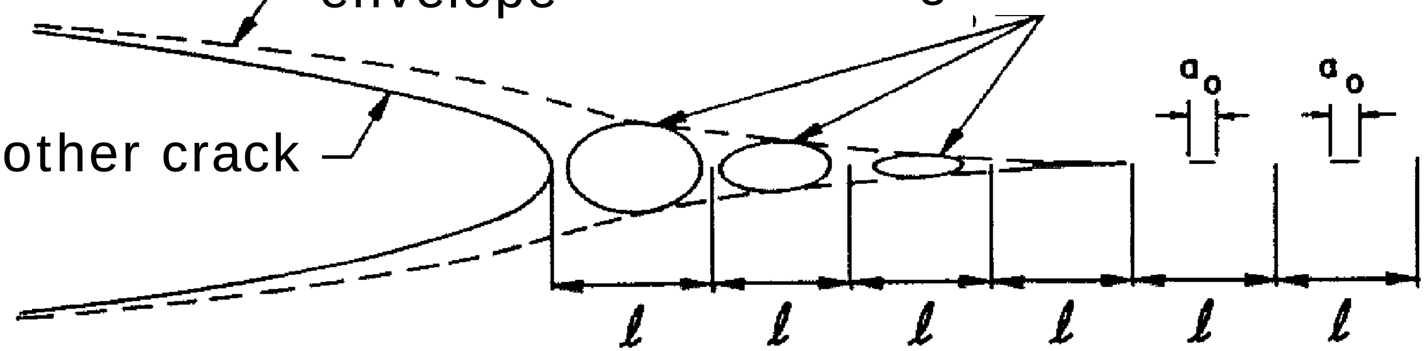
- Twin daughters (optimal by Jensen's inequality):

$$\Psi = n\alpha + n\left(\frac{v}{n}\right)^p \to \text{min} \Rightarrow$$

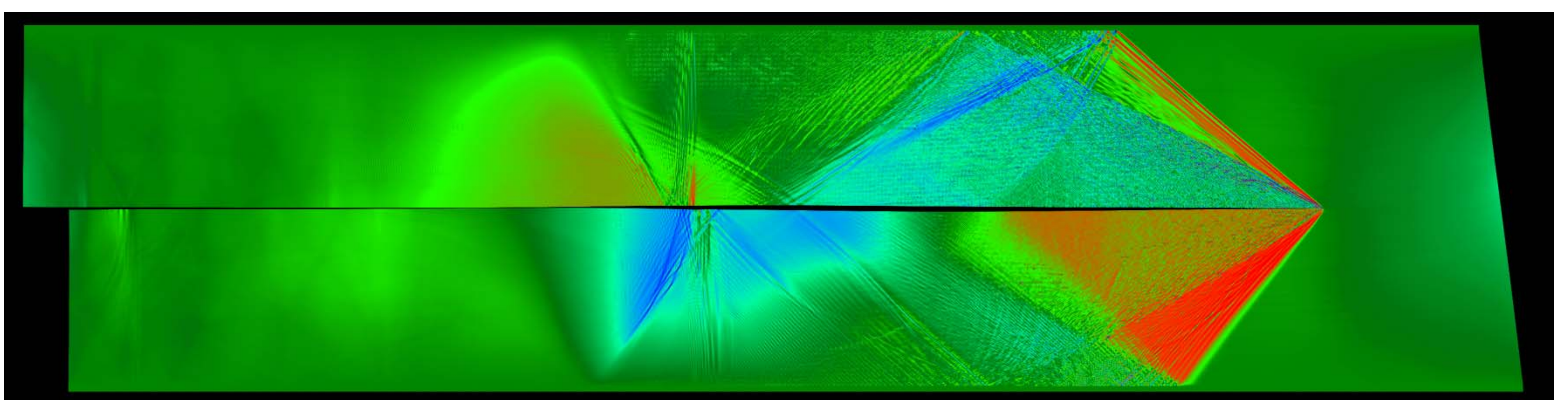
$$n_{\text{min}} = \left(\frac{p-1}{\alpha}\right)^{(1/p)} v, \quad \Psi_{\text{min}} = p\left(\frac{\alpha}{p-1}\right)^{(1-1/p)} v$$

LEFM energy-dissipation functionals

- The mother-daughter crack mechanism:



(Ortiz, *IJSS*, 1988)



(F. Abraham, M. Buehler, H. Gao...)

Concluding remarks

- Energy-dissipation functionals provide a useful tool for understanding microstructure evolution within the framework of the calculus of variations.
- They help to identify the ‘effective’ kinetics and energetics of systems that exhibit evolving microstructure
- Recovery sequences yield insight into microstructural evolution mechanisms
- Mother-daughter mechanism beats crack-front rate-dependency in fracture mechanics
- Causal limit?
- Inertia?