LAGRANGIAN FINITE ELEMENT ANALYSIS
OF NEWTONIAN FLUID FLOWS

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ABSTRACT

A fully Lagrangian finite element method for the analysis of Newtonian flows is developed. The approach furnishes, in effect, a Lagrangian implementation of the compressible Navier–Stokes equations. As the flow proceeds, the mesh is maintained undistorted through continuous and adaptive remeshing of the fluid mass. The principal advantage of the present approach lies in the treatment of boundary conditions at material surfaces such as free boundaries, fluid/fluid or fluid/solid interfaces. In contrast to Eulerian approaches, boundary conditions are enforced at material surfaces ab initio and therefore require no special attention. Consistent tangents are obtained for Lagrangian implicit analysis of a Newtonian fluid flow which may exhibit compressibility effects. The accuracy of the approach is assessed by comparison of the solution for a sloshing problem with existing numerical results and its versatility demonstrated through a simulation of wave breaking. The finite element mesh is maintained undistorted throughout the computation by recourse to frequent and adaptive remeshing © 1998 John Wiley & Sons, Ltd.

KEY WORDS: Lagrangian formulation; fluid flows; finite elements; mesh adaption; wave breaking

1. INTRODUCTION

The equations of motion Newtonian fluids are most commonly formulated in Eulerian form, leading to the Navier–Stokes equations. In many situations of interest, Eulerian formulations permit the simulation of fluid flows using a fixed domain or control volume, which is an attractive feature. There are cases, however, in which the conventional Eulerian approach is unduly cumbersome. Such is the case, for instance, of free-surface flows, flows involving interfaces between different species of fluids, and problems in which fluids are coupled to solids undergoing large deformations. The difficulty inherent to the Eulerian treatment of these problems has spawned a vast literature. Some attempts to overcome these difficulties have been the mark and cell (MAC) technique originally developed by Harlow and Welch¹ and ALE methods (e.g. References 2–4).

By contrast, the possibility of describing fluid flows within a fully Lagrangian framework, which renders the treatment of free surfaces and interfaces—as well as the compatibility between solids and fluids—trivial, has not correspondingly been explored, barring a few notable exceptions. The earliest fully Lagrangian treatment of viscous flows was seemingly advanced by Hirt et al.⁵ In

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their pioneering work, Belytschko et al. noted the advantages of the Lagrangian formulation for problems of fluid–structure interaction involving large structural deformations. Other noteworthy early works are those of Bach et al. and Ramaswamy et al. These approaches are based on a reformulation of the Navier–Stokes equations in material coordinates, which subsequently are updated via an *ad hoc* iterative procedure. However, because these methods were implemented on a fixed mesh, they were limited in the extent of the geometrical effects they could account for. As a notable exception, Kawahara et al. introduced a simple mesh rezoning technique and applied it to the simulation of solitary wave propagation.

In the past, the main technological obstacle standing in the way of fully Lagrangian finite element methods for fluids has been the lack of automated and adaptive meshing. Indeed, Lagrangian analyses based on a fixed mesh, such as carried out in the references listed above, inevitably lead to severe element distorsion soon after the inception of the flow and, consequently, are of limited scope. However, recent advances in meshing enable the continuous remeshing of the fluid mass as the flow proceeds, which effectively opens the way to the application of Lagrangian methods. The feasibility of the Lagrangian paradigm as it bears on unconstrained *solid* flows, such as occur in terminal ballistics and high-speed machining, has been amply established.

In the present paper, we develop a fully Lagrangian finite element method for the analysis of Newtonian flows based on continuous and adaptive remeshing. Our approach furnishes, in effect, a fully Lagrangian implementation of the compressible Navier–Stokes equations. The principal advantage of the present approach lies in the treatment of boundary conditions at material surfaces such as free boundaries, fluid/fluid or fluid/solid interfaces, specially where highly deformable solids are concerned. In contrast to Eulerian approaches, boundary conditions are enforced at material surfaces *ab initio* and therefore require no special attention. Expressions for the consistent tangents are provided as part of the outcome of the formulation, enabling fully implicit analysis with optimal equilibrium convergence. These consistent tangents are obtained by direct linearization of the incremental equations.

For simplicity, we restrict attention to two-dimensional flows. The accuracy of our approach is assessed by comparison of the solution for the sloshing problem with existing numerical results and its versatility demonstrated through a simulation of wave breaking. The ability of the method to follow simply the evolution of the surface profile as the wave breaks is noteworthy.

### 2. LAGRANGIAN DESCRIPTION OF NEWTONIAN FLOWS

Consider a body initially occupying a reference configuration $B_0$, and a motion

$$x_i = \phi_i(\mathbf{X}, t)$$

over $B_0$. Here, $\mathbf{X} \in B_0$ are the material or Lagrangian coordinates of a material particle, and $x$ the corresponding spatial or Eulerian co-ordinates at time $t$, as determined by the deformation mapping $\phi$. We consider a general initial value problem in the Lagrangian form

$$F_{ij} = \phi_{i,j} \quad \text{in } B_0$$

$$\phi_i = \bar{\phi}_i \quad \text{on } \partial B_{01}$$

$$P_{ij,j} + \rho_0 B_i = \rho_0 \bar{\phi}_i \quad \text{in } B_0$$

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\[ p_{iJ} N_J = \bar{T}_i \quad \text{on } \partial \Omega_2 \]  
\[ p_{iJ} = p_{iJ}(\mathbf{F}, \mathbf{F}) \quad \text{in } \Omega_0 \]  
\[ \phi_i(\mathbf{X}, 0) = \phi_i^{(0)}(\mathbf{X}) \quad \text{in } \Omega_0 \]  
\[ \phi_i(\mathbf{X}, 0) = V_i^{(0)}(\mathbf{X}) \quad \text{in } \Omega_0 \]

where \( F_{iJ} \) are the deformation gradients, \( \partial \Omega_{01} \) are the displacement boundary conditions on the reference displacement boundary \( \partial \Omega_{01} \), \( P_{iJ} \) is the Piola–Kirchhoff stress tensor, \( \rho_0 \) is the mass density in the reference configuration, \( B_i \) are the applied body forces per unit mass in the reference configuration, \( N_J \) is the outward unit normal to the reference boundary \( \partial \Omega_0 \), \( T_i \) are the traction boundary conditions on the reference traction boundary \( \partial \Omega_{02} \) and \( \sigma_{ij} \) are the initial displacement and velocity conditions. The Piola–Kirchhoff stress tensor \( P_{iJ} \) and the Cauchy stress tensor \( \sigma_{ij} \) are related as

\[ P_{iJ} = (J \sigma_{ij} P^{-1}_{iJ}) \circ \phi \]  
\[ J = \det \mathbf{F} \]

is the Jacobian of the deformation.

For a Newtonian fluid the constitutive equations are

\[ \sigma_{ij} = 2\mu d^\text{dev}_{ij} + p\delta_{ij} \]  
\[ d_{ij} = \frac{1}{2}(v_{i,j} + v_{j,i}) \]

with

\[ v_i = \dot{\phi}_i \circ \phi^{-1} \]

representing the components of the spatial velocity vector, and \( p \) is the pressure, which derives from an equation of state. For a compressible fluid we assume the pressure \( p \) to depend only on the volumetric part of the deformation, i.e.

\[ p = p(J) \]  
\[ J = 1 \]

To render these equations in finite element form, we begin by recasting (4) in the weak form

\[ \int_{\Omega_0} [P_{iJ} \eta_i - \rho_0 (B_i - \phi_i) \eta_i] \, d\Omega_0 - \int_{\partial \Omega_{02}} \bar{T}_i \eta_i \, dS_0 = 0 \]

where the test functions \( \eta_i \) satisfy the homogeneous essential boundary conditions

\[ \eta_i = 0 \quad \text{on } \partial \Omega_{01} \]
Pushing (16) forward to the spatial configuration leads to

$$
\int_B \left[ \sigma_{ij} \dot{\eta}_{i,j} - \rho (b_i - a_i) \dot{\eta}_i \right] \, dV - \int_{\partial B} \dot{t}_i \eta_i \, dS = 0
$$

(18)

where $B = \phi(B_0)$ and $\partial B_2 = \phi(\partial B_0)$ are the deformed domain and traction boundary, respectively. Finally, an application of (11) results in

$$
\int_B \left[ \sigma_{ij}^{\text{dev}} \dot{\eta}_{(i,j)} + p \, \text{div} \, \eta - \rho (b_i - a_i) \eta_i \right] \, dV - \int_{\partial B} \dot{t}_i \eta_i \, dS = 0
$$

(19)

which is the point of departure for the finite element implementation described in the sequel.

3. FINITE ELEMENT IMPLEMENTATION

3.1. Spatial discretization

Next, we seek to determine finite element interpolants of the form

$$
\phi_{ia}(X, t) = \sum_a x_{ia}(t) N_a(X)
$$

(20)

which approximate the exact motion $\phi_i(X, t)$. In (20), $x_{ia}(t) = \phi_{ia}(X_a, t)$ are the spatial nodal co-ordinates, $X_a$ are the material nodal co-ordinates, and $N_a$ are the undeformed or material shape functions. Inserting this representation into (18), a standard derivation leads to the semi-discrete equations

$$
\sum_b M_{iahb} \ddot{x}_{hb} + f_{ia}^{\text{int}}(x, \dot{x}) = f_{ia}^{\text{ext}}(t)
$$

(21)

where

$$
M_{iahb} = \sum e \int_{\Omega_e} \rho_0 \delta_{ik} N_a^e N_b^e \, d\Omega_0
$$

(22)

$$
f_{ia}^{\text{int}} = \sum e \int_{\Omega_e} P_i N_a^e \, d\Omega_0
$$

(23)

denote the consistent mass matrix and the internal force array, respectively, $\dot{x}$ is the acceleration array and $f^{\text{ext}}$ is the external force array resulting from the applied body forces and surface tractions.

We adopt the six-noded composite finite element devised by Camacho and Ortiz and mass lumping. It should be carefully noted that the element body forces, i.e. gravity forces, must be lumped accordingly in order for constant accelerations to result from the application of a constant force field, as required.

3.2. Temporal discretization

For the mass-dominated problems of interest here, such as the propagation of gravity waves, an implicit treatment of time is particularly attractive, if not mandatory. Consequently, we integrate

the system of non-linear ordinary equations (21) in time by recourse to Newmark’s algorithm (e.g. References 16,17)

\[ x_{n+1} = x_n + \Delta t v_n + \Delta t^2 \left( \frac{1}{2} - \beta \right) a_n + \beta a_{n+1} \]

\[ v_{n+1} = v_n + \Delta t (1 - \gamma) a_n + \gamma a_{n+1} \]

\[ M a_{n+1} + f^\text{int}(x_n, v_n) = f^\text{ext}_n \]

\[ M a_{n+1} + f^\text{int}(x_{n+1}, v_{n+1}) = f^\text{ext}_{n+1} \]

where \( x_n = x(t_n) \), \( v_n = \dot{x}(t_n) \), \( a_n = \ddot{x}(t_n) \), \( f^\text{ext}_n = f^\text{ext}(t_n) \), \( t_n = n\Delta t \) and \( 0 \leq \beta \leq 0.5 \), \( 0 \leq \gamma \leq 1 \) are the algorithm’s parameters. It is interesting to note that Newmark’s algorithm can be rephrased in the multistep form

\[ M \frac{x_{n+1} - 2x_n + x_{n-1}}{\Delta t^2} + a_{n-1}(f^\text{int} - f^\text{ext})_{n-1} + a_0(f^\text{int} - f^\text{ext})_n + a_1(f^\text{int} - f^\text{ext})_{n+1} = 0 \]

(25)

where the parameters \( (a_{-1}, a_0, a_1) \) are related to Newmark’s parameters \( (\beta, \gamma) \) as

\[ a_{-1} = \frac{1}{2} + \beta - \gamma, \quad a_0 = \frac{1}{2} - 2\beta + \gamma, \quad a_1 = \beta \]

(26)

The non-linear system of equations (24) may be solved by a Newton–Raphson iteration, leading to successive iterates \( (x^{(k)}_{n+1}, v^{(k)}_{n+1}, a^{(k)}_{n+1}) \), \( k = 0, \ldots, \). The starting iterate coincides with the Newmark predictor

\[ x^{(0)}_{n+1} = x_n + \Delta t v_n + \Delta t^2 \left( \frac{1}{2} - \beta \right) a_n \]

\[ v^{(0)}_{n+1} = v_n + \Delta t (1 - \gamma) a_n \]

\[ a^{(0)}_{n+1} = 0 \]

(27)

By this means, the system of nonlinear equations is reduced to

\[ x_{n+1} = x_{n+1}^{(0)} + \beta \Delta t^2 a_{n+1} \]

\[ v_{n+1} = v_{n+1}^{(0)} + \gamma \Delta t a_{n+1} \]

\[ M a_{n+1} + f^\text{int}(x_{n+1}, v_{n+1}) = f^\text{ext}_{n+1} \]

(28)

In order to compute the \( (k+1) \)th iterate, (28) is consistently linearized about the \( k \)th iterate leading to the equivalent linear static problem

\[ \left( \frac{K^{(k)}_{n+1}}{\beta} + \frac{\gamma}{\Delta t} C^{(k)}_{n+1} + \frac{1}{\beta \Delta t^2} M \right) u = f^\text{ext}_{n+1} - (f^\text{int})^{(k)}_{n+1} - M a^{(k)}_{n+1} \]

(29)

where \( u \) is the incremental displacement array and

\[ K = \frac{\partial f^\text{int}}{\partial x}(x, v), \quad C = \frac{\partial f^\text{int}}{\partial v}(x, v) \]

(30)
are the tangent stiffness and tangent damping matrices, respectively. A trite but straightforward calculation gives the consistent element tangents in the form

\[
K_{iakb} = \frac{\partial f_{e}}{\partial \dot{x}_{kb}} \int_{\Omega} \left[ \sigma_{im}(n_{b,k,h_{a,m}} - n_{b,m,h_{a,k}}) + \mu \left( v_{i,k,h_{b,j}}n_{a,j} - v_{j,k,h_{b,i}}n_{a,i} \right) ight. \\
+ \frac{2}{3} \left( v_{i,k,h_{b,i}}n_{a,i} - d_{b,k,h_{a,k}} \right) + \left. \frac{d}{dt} n_{b,k,h_{a,i}} \right] d\Omega
\]

and

\[
C_{iakb} = \frac{\partial f_{e}}{\partial \dot{x}_{kb}} = \int_{\Omega} \mu \left( \delta_{i,k,h_{b,m}}n_{a,m} + n_{b,m,h_{a,k}} - \frac{2}{3} n_{b,k,h_{a,i}} \right) dV
\]

where \( n_{a} \) are the deformed or spatial shape functions.

The precise form of these consistent tangents does not seem to have been derived before. Following the computation of \( u \), the nodal co-ordinates, velocities and accelerations are updated through an application of the Newmark correctors

\[
x_{n+1}^{(k+1)} = x_{n+1}^{(k)} + u
\]

\[
v_{n+1}^{(k+1)} = v_{n+1}^{(k)} + \frac{\gamma u}{\beta \Delta t}
\]

\[
a_{n+1}^{(k+1)} = a_{n+1}^{(k)} + \frac{u}{\beta \Delta t^2}
\]

The iteration terminates when a certain tolerance is met, e.g. when

\[
\frac{\| x_{n+1}^{(k+1)} - x_{n+1}^{(k)} \|}{\| x_{n+1}^{(1)} - x_{n+1}^{(0)} \|} \leq TOL
\]

Provided that the starting point of the iteration is within the radius of convergence of the solution, the use of the exact tangents (31) and (32) results in quadratic convergence.

4. ADAPTIVE MESHING

A difficulty which is encountered when Lagrangian methods are applied to problems involving unconstrained flow is the severe deformation-induced mesh distortion which is inevitably incurred. This difficulty can be effectively sidestepped by recourse to continuous and adaptive remeshing.13,15,18 Mesh adaption also furnishes an efficient means of resolving multiple scales in the solution with a minimum of degrees of freedom. An effective adaptive meshing capability must address two types of issues: representational, i.e. issues pertaining to the representation of domains; and analytical, including automatic meshing, adaption indicators and mesh-to-mesh transfer operators. A few salient issues among these are subsequently addressed in turn.

In calculations we represent domains as hierarchical systems.14,19–21 At the top level, the topological graph points to a set of bodies. A body may in turn comprise several subbodies, e.g. composed of different materials. The boundary of each subbody is decomposed into a collection of closed loops. The loops can be oriented consistently so as to unambiguously define the
interior and the exterior of the subbody. Each loop is partitioned into edges. An edge may appear, albeit with different orientations, in the boundaries of two different subbodies. Finally, the geometry of all edges is defined by piecewise quadratic interpolation from a collection of boundary nodes. The use of a sufficiently accurate interpolation of the boundary is essential in order to preserve the mass of the body.

4.1. The advancing front algorithm

The boundary representation of the model may be taken as a basis for the triangulation of the interior of the solids. The boundary representation serves the additional purpose of specifying the current contact surfaces in the solid. Advancing front methods constitute specially attractive automatic meshing techniques as they require a minimal set of input data—chiefly, the boundary information—from which both elements and interior nodes are generated simultaneously. Triangular elements are introduced one by one from the smallest segment on the front, which is taken as the base of the triangle. In the construction of the initial mesh, the size of the elements is determined by recourse to Jin and Wiberg’s control line technique. In this approach, the element size is computed as a weighted average of the boundary element sizes. In subsequent meshes, the size of the new elements inserted at the front is selected in accordance with a prespecified mesh density determined a posteriori from the solution. The advancing front algorithm terminates when no segments are left in the front. Carefully designed data structures and search algorithms need to be put in place to ensure an \(O(N \log N)\) operation count.

In our calculations we employ an \(h\)-adaptation strategy based on empirical refinement indicators to estimate the optimal mesh density, leading to simultaneous coarsening and refinement. The target mesh density is determined so as to equidistribute certain indicators over all elements in the mesh. As refinement indicator we adopt the bounded deformation norm of the velocity field which results in refinement (coarsening) in rapidly (slowly) varying regions of the flow. In our implementation, the element size information is interpolated on the old mesh, which serves as a background mesh for the advancing front algorithm.

4.2. Mesh-to-mesh transfer operator

The consistent formulation of transfer operators has been addressed by Ortiz and Quigley. The fundamental question to be ascertained concerns the formulation of consistent finite element equations when all fields at time \(t_n\) are supported on a mesh \(\mathcal{M}_n\) while the fields at time \(t_{n+1} = t_n + \Delta t\) are supported on a different mesh \(\mathcal{M}_{n+1}\). Ortiz and Quigley show that, when all finite element representations are introduced into the weak form of the equations of motion, the equilibrium and compatibility equations at \(t_{n+1}\) follow directly from the interpolation on \(\mathcal{M}_{n+1}\). Ortiz and Quigley further show that the weak form unambiguously determines the transfer operator.

In the context of a total displacement approach, where the reference configuration is fixed at the initial configuration, the deformation and velocity gradients and the Jacobian of the deformation can be computed directly from the nodal data. Since Newtonian fluids are history independent, i.e. the stresses depend solely on the current spatial velocity gradient and the Jacobian of the deformation (11), there is no need to transfer any state variables to the quadrature points of the new mesh. However, in problems of unconstrained flow, such as water waves propagation and breaking, an incremental formulation where the reference configuration is updated frequently is
advantageous, as the total deformation mapping nearly loses invertibility with the passage of time. In this case the accumulated Jacobian of the deformation must be transferred from mesh to mesh.

5. VALIDATION AND APPLICATIONS

5.1. Sloshing problems

By way of validation of the formulation just described we consider the simple problem of the free oscillation of an incompressible liquid in a container. Numerical results for this problem can be found elsewhere.\textsuperscript{2,8} Figure 1 depicts a schematic of the problem. Following Ramaswamy \textit{et al.}\textsuperscript{2,8} we consider a square liquid column of width $b$ with an initial surface profile corresponding to the first antisymmetric mode of vibration, i.e.

$$\eta(x, 0) = a \sin \frac{\pi x}{b}$$

where $a$ is the amplitude of oscillation. For the sake of comparison with previously published numerical results, we adopt a very small value for the amplitude ($a = 0.01$). In view of this small amplitude, the calculations can be carried out on a fixed mesh. The container walls are assumed to be impermeable and allow for free slip. A viscosity $\nu = 0.01$ is adopted in the calculations. Figure 2 shows the computed time histories of $\eta(\pm b/2, t)$ together with the same time histories reported by Ramaswamy \textit{et al.}\textsuperscript{8} As is evident from the figure, the two sets of results are in close agreement. It should be noted, however, that the geometric effects in this example are insignificant and that, in consequence, the full potential of the present method is not realized.

5.2. Wave breaking

In this section we present a simulation of the propagation of a water wave and its breaking due to shoaling over a plane slope. The geometry of the problem is shown in Figure 3. The inviscid problem has been analysed using boundary elements\textsuperscript{23} and the viscous case using ALE methods.\textsuperscript{2} The initial conditions are taken from the potential theory solution of a solitary wave of finite amplitude propagating without change of shape.\textsuperscript{23} In Laitone’s solution, which is frequently used
Figure 2. Time histories of surface elevation amplitude for the sloshing problem: (a) results from Ramaswamy et al.\textsuperscript{8}; (b) results for the present method.

Figure 3. Schematic of the analysis problem.
for comparison, the velocity, pressure and free surface elevation are:

\[ u = \sqrt{gd} \frac{H}{d} \operatorname{sech}^2 \left[ \sqrt{\frac{3H}{4d^3}}(x - ct) \right] \quad (35) \]

\[ v = \sqrt{3gd} \left( \frac{H}{d} \right)^{3/2} \frac{y}{d} \operatorname{sech}^2 \left[ \sqrt{\frac{3H}{4d^3}}(x - ct) \right] \tanh \left[ \sqrt{\frac{3H}{4d^3}}(x - ct) \right] \quad (36) \]

\[ \eta = d + H \operatorname{sech}^2 \left[ \sqrt{\frac{3H}{4d^3}}(x - ct) \right] \quad (37) \]

\[ p = \rho g (\eta - y) \quad (38) \]

\[ c = \sqrt{gd \left( 1 + \frac{H}{d} \right)} \quad (39) \]

As noted by Ramaswamy, Laitone’s solution holds for an infinitely long channel. However, for purposes of simulation it suffices to truncate the domain of analysis at a distance \( L/2 \) from the wave crest where the fluid is essentially still, e.g. where the surface elevation \( \eta = 0.01H \). The result is

\[ \frac{L}{d} = 6.9 \left( \frac{d}{H} \right)^{1/2} \]

In simulations, we take the constant depth region to be of length \( L \). The still water depth \( d \) is set to 10 m and the wave height \( H \) to 5 m. The slope of the shoaling bottom is set to 1/14. The acceleration of gravity is set to 9.8 m/s. The density of water is \( \rho = 1000 \text{kg/m}^3 \), the viscosity is \( \mu = 1.01 \times 10^{-3} \text{Ns/m}^2 \), and the compressibility modulus is \( K = 2.04 \times 10^9 \text{N/m}^2 \).

Plate 1 shows the initial adapted mesh and the velocity and pressure initial conditions. The initial mesh is adapted applying the refinement criterion presented in Section 4 to the initial velocity field. Plates 2–6 chronicle the evolution of the solution at times 4.0, 8.0, 9.0, 9.5 and 9.8, respectively. Four stages can be identified in the solution:

**Propagation:** The wave travels over the constant depth bottom towards the slope with no ostensible change of shape other than a small dispersion effect, Plate 2.

**Steepening:** As the wave hits the slope, a strongly non-linear behaviour becomes apparent. In particular, the crest of the wave accelerates while the trough lags behind. Eventually, the wave profile develops a vertical tangent and ceases to be a graph, Plate 3.

**Breaking:** Subsequently, the water jet formed at the crest plunges until it comes into contact with the nearly still surface of the water ahead, at which point the simulation is stopped, Plates 4–6.

The ability of the model to faithfully simulate the various stages of wave breaking is noteworthy. Good aspect ratios are maintained throughout the simulation. In addition, the mesh adaptively resolves the regions of rapid variation of the velocity field while remaining comparatively coarse in the quiescent regions, which tends to keep the number of degrees of freedom within reasonable bounds. The analysis requires 9800 implicit time steps of size \( \Delta t = 10^{-3} \text{s} \). A strong quadratic convergence is maintained throughout the simulation, with a maximum of three Newton–Raphson
iterations per time step required to attain a relative tolerance TOL of $10^{-13}$. The number of equilibrium iterations to convergence typically increases to as many as four immediately following a mesh adaption. This slow down in convergence is caused by the interpolation errors introduced by the mesh-to-mesh transfer operator.

Wave profiles at various time steps, shifted to the nominal (unperturbed) wave front center according to the nominal celerity (equation (39)), are presented in Figure 4.

6. SUMMARY AND CONCLUSIONS

A fully Lagrangian finite element method for the analysis of Newtonian flows based on continuous and adaptive remeshing has been developed. The method is specially advantageous for problems involving free fluid surfaces and interfaces; or the coupling of fluid flows and highly deformable solids. As the flow proceeds, the finite element mesh is maintained undistorted by recourse to continuous and adaptive remeshing. In addition, mesh adaption enables the simultaneous resolution of multiple scales in the solution. The versatility of the approach has been demonstrated through a simulation of wave breaking. The ability of the model to faithfully simulate the various stages of wave breaking is noteworthy.

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REFERENCES

Plate 1. Initial conditions and computational mesh
Plate 2. Solution fields and wave profile at time = 4.0 seconds
Plate 3. Solution fields and wave profile at time = 8.0 seconds
Plate 4. Solution fields and wave profile at time = 9.0 seconds
Plate 5. Solution fields and wave profile at time = 9.5 seconds
Plate 6. Solution fields and wave profile at time = 9.8 seconds