Adaptive Lagrangian modelling of ballistic penetration of metallic targets

G.T. Camacho, M. Ortiz*
Division of Engineering, Brown University, Providence, RI, USA

Received 23 May 1995

Abstract

A Lagrangian finite element model of ductile penetration is developed. Adaptive meshing is accorded a key role in following the large deformations which develop during penetration. An explicit contact/friction algorithm is used to treat the multi-body dynamics. Rate-dependent plasticity, heat conduction and thermal coupling are also accounted for in the calculations. The properties and predictive ability of the model are exhibited in several applications: copper rod impact, perforation of aluminum plates by conical-nosed projectiles and penetration of high-strength steel targets by WHA long rods. The simulations show close agreement with experimental observations and prior numerical results.

1. Introduction

The computational modelling of ballistic impact remains an active field of endeavor (see [1–3] for reviews). The computer codes which are used in the study of ballistic penetration can roughly be classified as Eulerian or Lagrangian, depending on whether the material flows through a fixed mesh or, by contradi distinction, the mesh follows the deformation of the solid. The advantages and disadvantages of each approach have been comprehensively reviewed by Anderson [4]. Eulerian codes, such as CTH [7] and HULL [8], have the appealing feature of using a fixed spatial grid. The formulation is tailored from the outset to flow-like motions and is not afflicted by deformation-induced mesh distortion. However, Eulerian formulations of ballistic penetration suffer from such shortcomings as: the need to advect material state histories; inherent difficulties in dealing with traction-free boundary conditions and contact between deformable bodies; the need for uniformly fine meshes to cover inactive—even empty—areas, with the concomitant increase in computer time and memory requirements; and, when the formulation is rigid-viscousplastic, inherent difficulties in following elastic unloading and computing residual stresses.

By way of contrast, the chief limitation of Lagrangian formulations based on a fixed mesh is that, in the presence of large deformations such as arise in ballistic penetration, finite elements may become severely distorted. This in turn may occasion the premature termination of the analysis, or, at the very least, result in unacceptably small time steps. Consequently, special measures need to be implemented to enable the analysis of unconstrained plastic flows by Lagrangian methods. For instance, EPIC [5] and DYNAP [6] make use of ‘eroding sliding interfaces’, whereby severely deformed or failed elements are removed—or ‘eroded’—in accordance with failure criteria such as critical accumulated plastic strain.

*Corresponding author.
A viable alternative is to resort to automatic, continuous and adaptive meshing. In the present context, mesh adaptivity entails the generation of meshes optimized so as to mitigate discretization errors and capture the steep gradients present in the solution. Additionally, by remeshing the deformed configuration, continuous remeshing has the beneficial effect of eliminating deformation-induced mesh distortions and, consequently, the need for deleting troublesome elements.

In this paper, we develop a Lagrangian finite element model of ballistic penetration which employs continuous adaptive remeshing in the manner just described. Critical elements of a successful adaptive meshing method include a versatile automatic mesh generator, efficient mesh refinement/coarsening strategies, and an accurate transfer operator. We describe these aspects of the method in Section 6. Our formulation is capable of treating multi-bodies and their interactions. Contact and friction is enforced between all potential contact surfaces, which are automatically determined. The contact algorithm is discussed in Section 3. Under severe impact conditions, the projectile and the target can undergo extensive plastic deformations and temperature rises. A sizeable fraction of the plastic work may be converted into heat, which is also generated at frictional contacts. The mechanical and thermal equations are fully coupled in consequence of thermal softening. The integration of the coupled thermo-mechanical equations is discussed in Section 4. Additionally, the pressures under the impactor can rise to very high values. Under these conditions, the conventional flow theories of plasticity have to be augmented by a suitable equation of state for the volumetric response. The constitutive framework adopted in calculations is described in Section 5. A related endeavor concerns the development of robust finite elements which retain adequate performance under extreme conditions of pressure and deformation. Matters of finite element design are briefly addressed in Section 2.

The capabilities of the model are demonstrated through the simulations presented in Sections 7-9. The problem of a cylindrical rod impacting a rigid surface provides a simple and convenient benchmark test for validating our numerical approach and investigating various mesh adaption trade-offs [55]. Adaptive meshing is found to afford significant speed-ups over fixed-mesh calculations by reducing the size of the problem and by ensuring good element aspect ratios, thus preventing the degradation of the time step required for stability. Applications to ballistic penetration of aluminum and steel targets are pursued in Sections 8 and 9. The results of the simulations are in excellent agreement with experiment. The simulations also reveal insights into the behavior of aluminum and steel at high strain rates and temperatures.

2. Finite element discretization

Consider a body initially occupying a reference configuration \( B_0 \), and a process of incremental loading whereby the deformation mapping over \( B_0 \) changes from \( \phi_x \) at time \( t_0 \) to \( \phi_{x+1} - \phi_x \cdot u \) at time \( t_{n+1} = t_n + \Delta t \). Dynamic equilibrium at time \( t_{n+1} \) is enforced weakly by recourse to the virtual work principle

\[
\int_{B_0} P_{n+1} : \nabla_{\theta} \eta \, dV_0 - \int_{B_0} (f_{n+1} - \rho_0 a_{n+1}) \cdot \eta \, dV_0 - \int_{\partial B_0} \eta \cdot t_{n+1} \, dS_0 = 0
\]  

(1)

where \( P_{n+1} \) denotes the first Piola–Kirchhoff stress field at time \( t_{n+1} \), \( f_{n+1} \), \( a_{n+1} \) and \( t_{n+1} \) are the corresponding body forces, accelerations and boundary tractions, respectively, \( \rho_0 \) is the mass density in the reference configuration, \( \eta \) is an admissible virtual displacement field, and \( \nabla_{\theta} \) denotes the material gradient. Upon a finite element discretization of (1), the governing equations become

\[
M a_{n+1} + F_{n+1}^{int} = F_{n+1}^{ext}
\]  

(2)

where \( M \) is a lumped mass matrix, \( F_{n+1}^{ext} \) is the external force array which accounts for the applied body forces and surface tractions, and \( F_{n+1}^{int} \) is the internal force array arising from the current state of stress.

The second-order accurate central difference scheme is used to discretize (2) in time [9-11], with the result
\[ d_{n+1} = d_n + \Delta t \cdot \mathbf{v}_n + \frac{1}{2} \Delta t^2 a_n \quad (3) \]

\[ a_{n+1} = M^{-1} \left( F_{n+1}^{\text{ext}} - F_{n+1}^{\text{int}} \right) \quad (4) \]

\[ \mathbf{v}_{n+1} = \mathbf{v}_n + \frac{1}{2} \Delta t (a_{n+1} + a_n) \quad (5) \]

where \( d, v \) and \( a \) denote the displacement, velocity, and acceleration arrays, respectively. Explicit integration is particularly attractive in impact problems, since the resolution of the various waves in the solution necessitates the use of small time steps well under the stability limit [12]. In addition, explicit contact algorithms are more robust and straightforward than their implicit counterparts, a distinct advantage in problems involving fragmentation where complicated contact situations inevitably arise. Explicit integration is also advantageous in three-dimensional calculations, where implicit schemes lead to system matrices which often exceed the available in-core storage capacity. Yet another desirable aspect of explicit algorithms is that they are ideally suited for concurrent computing [13].

The performance of an explicit dynamics formulation depends critically on the choice of element. The ability to construct meshes by triangulation simply and automatically provides an incentive for the use of triangular elements. First-order triangular elements, suffer from volumetric and shear locking which can result in gross inaccuracies [14], unless meshed in a cross-triangle configuration. However, this configuration is generally incompatible with the unstructured meshes considered here. By contrast, 6-noded elements with linear strain interpolation do not lock [9]. Unfortunately, off-the-shelf Lagrangian six-noded elements give rise to unbalanced midside and corner reactions bearing a 2:1 ratio in regular meshes subjected to uniform tractions, and the elements tend to perform poorly under severe impact conditions. Similar pathologies afflict the lumped mass matrix. To sidestep these difficulties, we use an assumed strain six-noded triangular element with linear stress and strain interpolation which is free of volumetric locking while yielding balanced midside and corner reactions and masses in the natural ratio of 1:1 [15]. This ensures, for instance, that a uniform acceleration is imparted to the nodes at the contact between a uniformly meshed straight-sided body and a flat rigid plate, a clearly desirable attribute.

3. Contact

In the applications of interest here, complicated contact situations develop which involve multiple collisions between deformable bodies. We have found the contact algorithm developed by Taylor and Banerjee [16] for the PRONTO2D explicit dynamics code particularly effective in dealing with such complex contact situations. In this approach, the bodies in contact can be deformable or rigid. The contacting surfaces are designated as master and slave. A balanced master-slave approach in which surfaces alternately act as master and slave during each half of the time step is employed. However, rigid surfaces are always treated as master surfaces.

The method starts with the calculation of predictor nodal positions, velocities and accelerations \( \mathbf{x}_{n+1}^{\text{pred}}, \mathbf{v}_{n+1}^{\text{pred}} \) and \( a_{n+1}^{\text{pred}} \), respectively, assuming that no contact has occurred. A predictor configuration where penetration has occurred is sketched in Fig. 1(a). In enforcing the contact conditions, it proves convenient to introduce an auxiliary consecutive numbering of the nodes on the contacting surfaces. The penetration distances \( \delta_{j,i} \) for all nodes \( j \) on the slave surface are then determined on the predictor configuration. Here and henceforth, labels \( m \) and \( s \) are used to designate the master and slave surfaces, respectively. The contact forces required to prevent penetration, were the master surface to remain stationary at the predictor configuration, are given by

\[ P_{s,i} = \frac{2M_{s,j} \delta_{s,j}}{\Delta t^2} \quad (6) \]

where \( M_{s,j} \) is the mass of node \( j \) on the slave surface. Next, normal acceleration corrections are introduced which eliminate the unwanted penetration, Fig. 1(b). The requisite accelerations are
\[ a_{\text{corr}}^{m} = \frac{\sum_{j} (\omega_{m-j} P_{s,j})}{M_{m,j} + \sum_{j} (\omega_{m-j} M_{s,j})}, \]
\[ a_{\text{corr}}^{s,j} = \sum_{k} (\omega_{m-s,k} a_{m,k}^{\text{corr}}) - \frac{P_{s,j}}{M_{s,j}} \]

where \( \omega_{m-j} \) and \( \omega_{m-s,k} \) are weights dependent on position. A Coulumb friction model is also adopted from [16]. Let \( t \) represent the unit tangent to the master segment. The tangential component of the relative predictor velocity between the slave node and the master segment is then given by

\[ \Delta v_{s,j} = t \cdot \left( \sum_{k} \omega_{m-s,k} v_{m,k}^{\text{pred}} \right) \]

The force that must be applied to the slave node to cancel its relative tangential velocity, i.e. to produce perfect stick, is

\[ F_{\text{Stick}}^{s,j} = -\frac{M_{s,j} \Delta v_{s,j}}{\Delta t} \]

The tangential force exerted by the master surface on a slave node cannot exceed the maximum frictional resistance

\[ F_{s,j} = \frac{F_{\text{Stick}}^{s,j}}{|F_{\text{Stick}}^{s,j}|} \min(\mu N_{s,j}, |F_{\text{Stick}}^{s,j}|, F_{\text{shearlim}}) \]

where \( F_{\text{shearlim}} \) accounts for the shear strength of the material and \( N_{s,j} \) is the normal contact force given by

\[ N_{s,j} = M_{s,j} a_{m,j}^{\text{corr}} \cdot n \]

Here, \( n \) is the unit normal to the surface. The tangential force induces the tangential acceleration corrections

\[ a_{t,j}^{\text{corr}} = \frac{F_{s,j}}{M_{s,j}} \]
\[ a_{m,j}^{\text{corr}} = \frac{\sum_{j} (\omega_{m-j} F_{s,j})}{M_{m,j}} \]

This completes the computation of the acceleration array. The corresponding displacement and velocity corrections follow from (3) and (5).

In implementing a contact algorithm, careful attention has to be paid to the contact search and log
in order to ensure computational efficiency and correctness. These issues have been addressed in detail elsewhere (e.g. [16,17]).

4. Thermal effects

In the course of an impact event, substantial amounts of heat may be generated due to the plastic working of the solid and friction at the contact surfaces. The temperatures attained can be quite high and have a considerable influence on the mechanical response. The relevant balance law to be considered in this case is the first law, which can be expressed in weak form as

\[
\int_{\Omega} \rho c T \eta \, dV + \int_{\partial \Omega} h \eta \, dS = \int_{\Omega} q \cdot \eta \, dV + \int_{\partial \Omega} s \eta \, dV
\]  

(15)

where \( \rho \) is the current mass density, \( c \) the heat capacity, \( T \) the spatial temperature field, \( \eta \) an admissible virtual temperature field, \( h \) the outward heat flux through the surface, \( q \) is the heat flux, \( s \) is the distributed heat source density and \( B_{\eta} \) the outward Neumann boundary. The main sources of heat in our applications are plastic deformation in the bulk and frictional sliding at the interface. The rate of heat supply due to the first is estimated as

\[
s = \beta \dot{W}^p
\]  

(16)

where \( \dot{W}^p \) is the plastic power per unit deformed volume and \( \beta \) is the Taylor–Quinney coefficient [18,19]. The rate at which heat is generated at the frictional contact, on the other hand, is

\[
h = -t : [\dot{\nu}]
\]  

(17)

where \( t \) is the contact traction and \([\dot{\nu}]\) is the jump in velocity across the contact. This heat must be apportioned between the bodies. Using transient half-space solutions, the ratio of the heat supply to body 1, \( h_1 \), and body 2, \( h_2 \), can be computed as [20,23]

\[
\frac{h_1}{h_2} = \frac{\sqrt{k_1 \rho_1 c_1}}{\sqrt{k_2 \rho_2 c_2}}
\]  

(18)

where \( k_\alpha, \rho_\alpha \) and \( c_\alpha, \alpha = 1, 2, \) are the thermal conductivity, mass density and heat capacity of the contacting bodies.

Inserting the finite element interpolation into (15) results in the semi-discrete system of equations [11]

\[
CT + KT = Q
\]  

(19)

where \( T \) is the array of nodal temperatures, \( C \) is the heat capacity matrix, \( K \) is the conductivity matrix, and \( Q \) is the heat source array. In the applications of interest here, the mechanical equations always set the critical time step for stability. It therefore suffices to lump the capacitance matrix and integrate the energy equation (19) explicitly by the forward Euler algorithm [9,11], with the result

\[
T_{n+1} = T_n + \Delta t \cdot \dot{T}_n
\]  

(20)

\[
CT_n + K T_n = Q_n
\]  

(21)

A staggered procedure [21] is adopted for the purpose of coupling the thermal and mechanical equations. Mechanical and thermal computations are staggered assuming constant temperature during the mechanical step and constant heat generation during the thermal step. Following Lemonds and Needleman [22], a mechanical step is taken first based on the current distribution of temperatures, and the heat generated is computed from (16) and (17). The heat thus computed is used in the thermal analysis where temperatures are recomputed by recourse to the forward-Euler algorithm (20) and (21). The resulting temperatures are then used in the mechanical step and incorporated into the thermal-
softening model described in Section 5, which completes one time-stepping cycle. A flowchart of the staggered procedure is shown in Box 1.

5. Constitutive model

Constitutive equations suited to the conditions which arise in ballistic penetration can be formulated by assuming that the volumetric or dilatational response is governed by an equation of state while the shear or deviatoric response obeys a conventional flow theory of plasticity. We begin by decomposing the Cauchy stress tensor into hydrostatic and deviatoric components,

\[ \sigma_{ij} = -p \delta_{ij} + s_{ij}, \]  

where \( p \) is the hydrostatic pressure and \( s_{ij} \) is the stress deviator. We adopt an equation of state of the Mie-Grüneisen type,

\[ p = (K_1 \mu + K_2 \mu^2 + K_3 \mu^3) \left( 1 - \frac{1}{2} \Gamma u \right) + \Gamma p e, \]  

where \( \mu = J^{-1} - 1 \), \( J = \text{det} \mathbf{F} = \rho \mu \) is the Jacobian of the deformation, \( \mathbf{F} \) is the deformation gradient, \( \rho \) is the current mass density, \( e \) is the internal energy per unit mass, \( \Gamma \) is the Grüneisen coefficient and \( K_1, K_2 \) and \( K_3 \) are material coefficients. In addition, a small artificial bulk viscosity is introduced to prevent high velocity gradients from collapsing elements and to damp down ringing [16]. The viscous pressure takes the form

\[ q = b_1 \rho e c_d \left( \frac{f}{J} \right) - \rho \left( b_2 \frac{f}{J} \right)^2 \]  

where \( b_1 \) and \( b_2 \) are constants, \( c_d \) is the dilatational wave speed, \( f/J \) is the volumetric strain rate and \( / \) is a typical element dimension. The principle of conservation of energy requires the increase in internal energy of any subbody to equal to the sum of the work of deformation and the heat input into the subbody, leading to the identity

\[ p e = \sigma_{ij} d_{ij} + \rho e i \]  

where \( d_{ij} = (d_{i,j} + d_{j,i})/2 \) is the rate of deformation tensor. In calculations, we update the pressure and the internal energy by recourse to the integration scheme developed in [16].

For the deviatoric response we adopt a standard formulation of finite deformation plasticity based on a multiplicative decomposition of the deformation gradient into elastic and plastic components. We employ the fully implicit algorithm of Catté and Ortiz [24] for performing the constitutive update. The governing constitutive equations and their discretized counterparts can be written as

Constitutive ↔ Incremental:

\[ \mathbf{F} = \mathbf{F}^p \mathbf{F}^o \leftrightarrow \mathbf{F}_{n+1} = \mathbf{F}_n^p \mathbf{F}_{n+1}^o \]  

\[ \mathbf{F}^o \mathbf{F}^o = \mathbf{e}^o \mathbf{R}(\mathbf{S}, \mathbf{Q}) \leftrightarrow \mathbf{F}_{n+1}^o = \exp(\mathbf{e}^o \mathbf{R}_{n+1}) \mathbf{F}_n^o \]
\[ S = S \left( \frac{1}{2} \log C^e \right) \rightarrow S_{n+1} = S \left( \frac{1}{2} \log C_{n+1}^e \right) \]  
\[ \tilde{\theta} = \dot{\varepsilon}^p \tilde{H}(\tilde{\varepsilon}, \tilde{\sigma}) \rightarrow \tilde{\theta}_{n+1} = \tilde{\theta}_n + \Delta \varepsilon^p \tilde{H}_{n+1} \]  
\[ \varepsilon^p = \frac{1}{\eta} \dot{\phi}(\tilde{S}, \tilde{\sigma}) \rightarrow \Delta \varepsilon^p = \frac{\Delta t}{\eta} \phi_{n+1} \]  

Eq. (26) embodies the assumption that the deformation gradient \( F \) admits a multiplicative decomposition into elastic and plastic parts, \( F^e \) and \( F^p \), respectively. The plastic part \( F^p \) of the deformation gradient field defines an additional (generally incompatible) configuration \( B^p \), or intermediate configuration. We shall use an overbar to identify fields defined over the intermediate configuration. Eq. (27) is the plastic flow rule which, as stated here, determines \( F^p \) fully. The manner in which the flow rule is discretized is an essential part of the method, and consists of taking the plastic multiplier \( \tilde{\varepsilon} \) to be constant throughout the increment and equal to its final value \( \tilde{\varepsilon}_{n+1} \). This reduces the flow rule to a system of linear equations for \( F^p \) with initial conditions \( F^p_n \), the exact solution of which is given by the exponential mapping in (27).

Without loss of generality, we express the elastic response (28) in terms of logarithmic elastic strains \((1/2) \log C^e\), where \( C^e = F^e \cdot F^e \) is the elastic right Cauchy–Green deformation tensor. The stress measure \( S \) in (28) is the second Piola–Kirchhoff stress tensor relative to the intermediate configuration. The symbol \( \tilde{\sigma} \) denotes some suitable set of internal variables referred to the intermediate configuration, with \( \tilde{\sigma} \) the corresponding hardening modulus. Finally, \( \varepsilon^p \) is an effective plastic strain, \( \phi \) is an effective overstress and \( \eta \) a viscosity parameter. We remark that, by expressing the flow rule, elastic response and hardening laws on the intermediate configuration, the formulation automatically satisfies material frame indifference. Eqs. (26–30) define a system of nonlinear equations which, for given \( F_{n+1} \), can be solved for the updated state variables \( S_{n+1}, \tilde{\sigma}_{n+1}, \tilde{\varepsilon}^p_{n+1} \), as well as for \( \Delta \varepsilon^p \). As noted by Cuitino and Ortiz [35], it is possible to reduce the system to a single equation for \( \Delta \varepsilon^p \), which can be solved by recourse to a local Newton–Raphson iteration. Cuitino and Ortiz [35] have also noted that the update defined in the foregoing furnishes a material-independent extension of small-strain updates into the finite deformation range.

In a typical ballistic penetration event, very high strain rates may be attained at which the flow stress may exhibit a strong rate-sensitivity [25–27]. In our calculations we employ the conventional power law

\[ \varepsilon^p = \varepsilon_0^p \left( \frac{\tilde{\sigma}}{g(\varepsilon^p)} \right)^m, \quad \text{if } \tilde{\sigma} > g(\varepsilon^p) \]  
\[ \varepsilon^p = 0, \quad \text{otherwise} \]  

where \( \tilde{\sigma} \) is the effective Mises stress, \( g \) the flow stress, \( \varepsilon_0^p \) the accumulated plastic strain, \( \varepsilon_0^p \) a reference plastic strain rate, and \( m \) the strain rate sensitivity exponent. Additionally, we adopt an equally conventional power hardening law in conjunction with Johnson and Cook's [64] power thermal softening law, which gives

\[ g = \sigma_0 \left[ 1 - \left( \frac{T - T_m}{T_o - T_m} \right)^a \right] \left( 1 + \frac{\varepsilon^p}{\varepsilon^p_0} \right)^{1/n} \]  

where \( n \) is the hardening exponent, \( \varepsilon^p_0 \) is a reference plastic strain, \( T \) the current temperature, \( T_0 \) a reference temperature, \( T_m \) the melting temperature, \( a \) the thermal softening exponent, and \( \sigma_0 \) is the yield stress at \( T_0 \). It should be noted that, owing to the staggered integration of the coupled thermomechanical equations, the temperature \( T \) remains fixed during a mechanical step and, therefore, plays the role of a known parameter during a stress update.

## Adaptive meshing

Adaptive meshing is increasingly being recognized as a crucial part of finite element modelling in such areas as application as manufacturing, crushworthiness, and, of primary interest to us here, ballistic
penetration, to mention a few salient examples [28–30]. The advantages of adaptive methods are many, including the potential for a drastic reduction in the number of degrees of freedom required to achieve a prescribed level of accuracy or resolution [30], especially in three dimensions. In solid mechanics applications, adaptive mesh refinement provides a powerful tool for systematically increasing the accuracy of solutions in linear problems (e.g. [46]) and for alleviating element distortion and resolving sharp gradients in finite deformation problems [20, 31, 33, 51]. Continuous adaptive remeshing enables the simulation of unconstrained plastic flows within a Lagrangian framework and provides an attractive alternative to erosion methods in ballistic penetration calculations.

A method of adaption begins with a model of the solid to be analyzed. The model contains relevant boundary information such as is required to automatically generate a first mesh over the solid. As the solution proceeds, the mesh follows the deforming boundary. In addition, suitable remeshing indicators provide a basis for refining and coarsening the mesh in the interior of the solid so as to meet accuracy or resolution criteria. In the analysis of inelastic solids, the state variables need to be transferred by some suitable means to the new mesh following each adaption. These aspects of the formulation are next discussed in turn.

6.1. System representation

In calculations where bodies are hierarchical systems [17, 38–40], Fig. 2. At the top level, the hierarchy comprises a set of bodies. A body in turn comprises a boundary and an interior. In a two-dimensional context, each body is regarded as a patch bounded by boundary curves defined by control nodes. The boundary of the patch is composed of both exterior and interior surfaces. The latter may arise, e.g. as cracks nucleate in the interior of the body. The boundary is discretized into edges containing boundary nodes. Finally, the interior of the body is discretized into finite elements. An example of a system defined in this manner is shown in Fig. 3. A carefully designed solid-modelling procedure is essential in applications involving many evolving and interacting bodies, such as develop in the course of fragmentation simulations [1/1].

6.2. Mesh generation

A number of automatic mesh generation schemes have been proposed including Delaunay triangulation [41], quadtree/octree methods [42], advancing front triangulation [40] and paving [43]. Advancing front methods are attractive since they require a minimal set of input data—primarily, the boundary information composed of boundary segments and nodes—from which both elements and interior nodes are generated simultaneously. A sample mesh generated by the advancing front method is shown in Fig. 4. In addition, advancing front methods offer great flexibility for mesh adaption. In our calculations we employ a refinement indicator to determine the optimal mesh density, leading to simultaneous coarsening and refinement.

Triangular elements are introduced one by one from the smallest segment on the front, which is

---

**SYSTEM**

- Bodies

**BOUNDARY**

- Control Nodes Curves
- Boundary Nodes Discrete Edges

**INTERIOR**

- Patches
- Interior Nodes Finite Elements

**GEOMETRY DISCRETIZATION**

Fig. 2 System hierarchy
taken as the base of the triangle. In the construction of the initial mesh, the size of the elements is determined by recourse to Jin and Wiberg's [39] control line technique. In this approach, the element size is computed from the weighted lengths of the nearest faces on the boundary. These faces, which form what we term 'control lines', constitute an integral part of the solid-modelling hierarchical system described earlier and, in consequence, are maintained and updated throughout the calculation. The control lines serve the dual purpose of specifying the current contact surfaces in the solid. An alternative method of element size control has been advanced by Peraire et al. [40] and relies on the use of a background triangular mesh. Metric information such as element sizes, stretches and orientations is stipulated at the nodes of the background mesh, and the metric properties of specific elements are determined by interpolation. We adopt this strategy for the purpose of remeshing, with the background mesh identified with the old mesh.

Once the element size is known, the third vertex which, in conjunction with the active front segment, defines the new element is either introduced or identified with an existing point on the front. This latter
case usually results in the removal of segments from the front. If a new node is introduced, its position is chosen so as to obtain an acceptable aspect ratio while avoiding intersections with other segments on the front [40]. The front is then updated by deleting the active segment and adding the sides of the newly defined element. The process ends when no segments are left in the front. In the example shown in Fig. 4, the control lines are identified with the straight sides composing the exterior boundary and the interior circles. The exterior (interior) control lines are discretized coarsely (finely), which results in a coarse-to-fine gradation of element sizes.

For a mesh of size $N$, a naive implementation of the advancing front method can lead to an $O(N^2)$ operation count. The more burdensome operations are [44]: finding the next face to be deleted; finding the nodes closest to a given point; finding the faces adjacent to a given point; and interpolating metric information from an unstructured background mesh. To ensure an $O(N \log N)$ algorithm we adopt the data structures and operations suggested by Lohner [44]. These are: quadtree structures to store nodes; heap lists to store faces; and link lists to store node-face relations. A summary of the mesh generation procedure is collected in Box 2.

6.3. Mesh adaption

While adaptive meshing is generally regarded as an indispensable tool for computing high-quality solutions with an economy of means, views on how best to achieve this goal vary. The prevailing adaptive meshing paradigms can be roughly classified as $r$, $h$, $p$ and $h$-$p$. The $r$-paradigm (e.g., [45]) or moving grid method, seeks to maintain the number of elements and nodes fixed while adapting the nodal positions. The method is simple to implement but is limited in power and flexibility. The $h$ paradigm (e.g., [46,47]) keeps the order of the elements unchanged while seeking to improve the solution by adaptive mesh refinement and coarsening. In the $p$-approach (e.g., [48]), the number of elements is held fixed and their order increased or decreased adaptively in accordance with accuracy requirements. A combination of $h$ and $p$ adaptivity is used in $h$-$p$ methods [48].

In our calculations we employ an $h$-adaption strategy with meshes generated automatically by the advancing front method described in the foregoing. We begin by saving the nodal and element data for subsequent transfer to the new mesh. Each body in the model is processed in turn. Adaption is triggered by the attainment of threshold values of certain quantities such as element distortion measures, element and nodal indicators, and contact surface compatibility measures. We use the ratio $IR/CR$ of the radius $IR$ of the inscribed circle to the radius $CR$ of the circumscribed circle as our measure of element distortion, where

$$IR = \frac{\sqrt{s(s-h_1)(s-h_2)(s-h_3)}}{s}$$

$$CR = \frac{\frac{h_1h_2h_3}{4\sqrt{s(s-h_1)(s-h_2)(s-h_3)}}}$$

Box 2:

Mesh generation procedure

(i) Define the boundary control lines.

(ii) Discretize the control lines into nodes and faces. Construct a quadtree of nodes, a heap list of faces and a link list of nodes and faces.

(iii) Extract the smallest face on the front from the heap list. Determine the triangle size and a tentative position for the new node. From the quadtree, decide whether to accept the node or to use an existing node in the front. Form the new triangle and update the front.

(iv) Update the quadtree of nodes, heap list of faces and link list of nodes and faces.

If front is empty GOTO (v).

Otherwise GOTO (iii).

(v) Apply Laplacian smoothing.

(vi) Add midpoint nodes.
and \( h_i \), \( i = 1, 2, 3 \) are the lengths of the triangle sides and \( s = (h_1 + h_2 + h_3) / 2 \). For an equilateral triangle, \( IR / CR = 1 / 2 \). At the opposite extreme, \( IR / CR \) tends to zero as the triangle becomes narrower. Remeshing is deemed necessary when \( IR / CR \) falls below a prescribed tolerance in some element. The distortion of the boundary is also monitored through the approximate curvature measure

\[
\kappa = \frac{\Delta \theta}{\Delta s}
\]

(36)

where \( \Delta \theta \) is the angle made by the two segments of a six-noded element side and \( \Delta s \) is the side length. The element is targeted for refinement when \( \kappa \) exceeds a prescribed tolerance.

The adaption of the interior mesh of the body is driven by an interpolation error measure in the spirit of Diaz et al. \[45\]. These indicators have the appeal of providing a direct measure of the ability of the mesh to resolve the fine local structure of the solution. In particular, they do not rely on elliptic error estimates, which are of limited applicability beyond linear elasticity and a restricted class of model problems. In their simulation of shear bands, Ortiz and Quigley \[51\] proposed the bounded variation norm of the element velocity field

\[
I_e = \left\| \mathbf{v}_e \right\|_{BV}
\]

(37)

as the refinement indicator \( I_e \). By this criterion, elements in which the variation of the velocity field exceeds a preset tolerance are deemed too coarse to resolve the local structure of the solution and are targeted for refinement. The advantage of indicator (37) is that it applies to a very general class of functions which may contain strong discontinuities, or ‘shocks’, over sets of finite perimeter. Velocity fields in this class indeed arise in some models of localization \[34\]. If the velocity field is smooth, or belongs to the Sobolev space \( W^{1,1} \), the indicator (37) is equivalent to

\[
I_e = \int_{\Omega_e} \sqrt{\frac{1}{2} d_1 d_2}
\]

(38)

which has been used by Batra and Ko \[49\] in localization calculations, and by Chen and Batra \[50\] to simulate the penetration of aluminum targets by rigid projectiles, and is adopted in the calculations reported here. Other adaption indicators may be better suited to other applications. For instance, the maximum difference in effective plastic strain over the sides of an element has been used by Bachmann et al. \[53\] in metal forming simulations; the plastic work in an element has been used by Coutinho and Ortiz \[35\] to simulate the evolution of slip bands in ductile single crystals; and the plastic work rate in an element has been employed by Marusich and Ortiz \[20\] in simulations of high-speed machining. The plastic work rate criterion and the bounded deformation criterion (38) are roughly equivalent if the material is weakly hardening, and identical if the material is ideally plastic.

The total interpolation error for the mesh is

\[
\mathcal{J} = \sum_{e=1}^{n_{el}} I_e
\]

(39)

where the sum extends to all \( n_{el} \) elements in the mesh. The objective is now to design the mesh so as to minimize \( \mathcal{J} \) for a given number \( n_{el} \) of elements. The optimality condition was found by Devloo et al. \[36\] and Oden et al. \[37\] to be the equidistribution of \( I_e \) over the elements of the mesh. In order to effect this equidistribution, we begin by computing the average of \( I_e \) over the old mesh as

\[
I_{ave} = \frac{1}{n_{el}} \sum_{e=1}^{n_{el}} I_e = \frac{\mathcal{J}}{n_{el}}
\]

(40)

A normalized adaption indicator is defined as

\[
\xi_e = \frac{I_e}{I_{ave}}
\]

(41)

In two dimensions, the target size of the element is set to
\[ h_e = \frac{h_e^{\text{old}}}{\xi_e} \quad (42) \]

Evidently, if \( l_e > l_{\text{ave}} \), then \( \xi_e > 1 \) and \( h_e < h_e^{\text{old}} \), leading to refinement of element \( e \). Contrariwise, if \( l_e < l_{\text{ave}} \), then \( \xi_e < 1 \) and \( h_e > h_e^{\text{old}} \), leading to coarsening. To avoid undesirably large or small elements, we constrain the element size to lie within prescribed bounds \( h_{\text{min}} \) and \( h_{\text{max}} \), i.e.

\[ h_{\text{min}} \leq h_e \leq h_{\text{max}} \quad (43) \]

The main reason for resorting to adaptive coarsening is to prevent an excessive proliferation of elements resulting in runaway problem sizes. In calculations, we append to the adaption criterion various empirical rules for inhibiting coarsening in areas where state history information should not be lost, e.g. in regions where the effective plastic deformation exceeds a threshold value.

In the context of the advancing front method, the element size information is reduced to the nodes of the background mesh, which we identify with the old mesh. This reduction is effected by averaging the sizes \( h_e \) of the \( n_a \) elements connected to node \( a \), with the result

\[ h_a = \frac{1}{n_a} \sum_{e=1}^{n_a} h_e \quad (44) \]

and \( h_a \) is taken as the nodal value of the distribution of element sizes. Surface distortion is accounted for by halving element sizes where a threshold value of the mean curvature (36) is exceeded, which has the effect of inducing refinement around sharp corners. Mesh adaption additionally furnishes convenient means of ensuring that contact conditions are accurately accounted for. For instance, gross inaccuracies may arise when a sharp corner comes in contact with a coarsely meshed surface. Following Bachmann et al. [53], we remedy these situations by matching element sizes across contact surfaces. This completes the determination of target element sizes at the nodes of the background mesh.

The discretization of the interior of the bodies now proceeds as described in Section 6.2. The initial front follows by discretization of the boundary in accordance with the element size distribution. The discretization of the boundary is effected one control line at a time. Each three-node boundary segment is interpolated quadratically leading to a piecewise quadratic representation of the control lines. The use of a sufficiently accurate interpolation of the boundary is essential in order to preserve the mass of the body. Following Peiró et al. [52] the number of boundary segments is determined as the closest integer \( N_e \) to

\[ A_e = \int_0^{l_e} \frac{1}{h(s)} \, ds \quad (45) \]

where \( s \) is the arc-length measured along the boundary and \( l_e \) is the length of the control line. The arc-length of new node \( a \) then follows from the equation

\[ \frac{N_e}{A_e} \int_0^{l_e} \frac{1}{h(s)} \, ds = a , \quad a = 0, 1, \ldots, N_e \quad (46) \]

In these expressions, \( h(s) \) is interpolated linearly in \( s \) from its nodal values \( h_a \) on the boundary of the background mesh. Care has to be exercised not to round off sharp corners which may arise as a result of the deformation. Examples of deformation-induced corners may be found in the examples reported in Section 9. The emergence of corners is detected by monitoring the angles subtended by boundary segments. When the angle at a boundary node exceeds a preset threshold value, the node is added to the control node set and the control line containing it is partitioned into two control lines. The piecewise quadratic interpolation of the boundary may also introduce small interpenetrations of the contact surfaces which, left uncorrected, may result in spurious accelerations. However, these interpenetrations can be eliminated simply through small adjustments of the nodal positions. These adjustments result in negligible mass changes in the bodies. The complete adaption procedure is summarized in Box 3.
5.4 Transfer operators

When history-dependent materials are involved, the continuation of the calculations after remeshing requires the transfer to the new mesh of the state variable data. The consistent formulation of transfer operators has been addressed by Ortiz and Quigley [51]. The fundamental question to be ascertained concerns the formulation of consistent finite element equations when all fields at time $t_n$ are supported on a mesh $\mathcal{M}_n$, while the fields at time $t_{n+1} = t_n + \Delta t$ are supported on a different mesh $\mathcal{M}_{n+1}$. Ortiz and Quigley [51] show that, when all finite element representations are introduced into the Hu–Washizu variational principle, the equilibrium and compatibility equations at $t_{n+1}$ follow directly from the interpolation on $\mathcal{M}_{n+1}$. By contrast, the constitutive update giving the state variables on mesh $\mathcal{M}_{n+1}$ is found to comprise two steps. The first step consists of a mapping of the state variables from $\mathcal{M}_n$ onto $\mathcal{M}_{n+1}$. The second step is a conventional state update on $\mathcal{M}_{n+1}$ based on the transferred state variables. Ortiz and Quigley [51] further show that, once the interpolation of the state variables is decided upon, the Hu–Washizu principle precisely determines the transfer operator.

Ortiz and Quigley’s [51] method provides a convenient and systematic tool for formulating variationally consistent transfer operators. However, a number of matters of implementation need to be addressed on a more empirical basis. For instance, numerical noise may have to be filtered out of the state data prior to the transfer. If the solids obey conventional $J_2$-flow theory of plasticity, the deformation field is nearly incompressible in the fully developed plastic regime. However, the variationally consistent transfer operators, in consequence of their linearity, do not commute with the mapping $J = \det(F)$. Therefore, if the transfer operator is applied to the deformation gradient $F$ directly, the transferred field may not be volume preserving even if $J = 1$, say, on all quadrature points of the old mesh. In cases in which the volumetric response of the solid is stiff, this may result in large spurious pressures. An additional source of difficulty arises when rate-sensitivity is modelled by a conventional power law such as (31). If the rate-sensitivity exponent is very large, representing nearly inviscid behavior, small variations in the effective Mises stress introduced by the transfer operator may result in large spurious plastic strain rates.

For the constitutive model formulated in section 5, a complete set of functionally independent state variables is the deformation gradient $F$, the plastic deformation gradient $F^p$, the effective plastic strain $\varepsilon$, and the internal energy $e$. We collectively denote the set of internal variables $\Lambda$. However, as pointed out earlier the transfer of the set $\Lambda = \{F, F^p, \varepsilon, e\}$ does not commute with the determinant mapping $J = \det(F)$, $J^p = \det(F^p)$, which may result in gross departures from plastic isochoricity. To overcome this difficulty, we choose instead the equivalent set $\Lambda = \{\log U, \log R, \log U^p, \log R^p, \varepsilon, e\}$, where $F = RU$ and $F^p = R^pU^p$ are the polar decompositions of $F$ and $F^p$, respectively. Volume preservation is now ensured by the linearity of the volumetric deformation mapping $J = \text{trace}(\log U)$, which commutes with the likewise linear transfer operator. In particular, the plastic
isochoricity condition \( \log J^p = 0 \) is exactly preserved by the transfer. In calculations, we compute \( \log U \) (\( \log U^p \)) from a spectral decomposition of \( C = F^T F \) (\( C^p = F^p Q F^p \)), which is readily effected in two dimensions. Explicit expressions for \( \log R \) have been given, e.g. by Moham et al. [54]. Following transfer, the deformation gradients are recovered by the exponential formulae

\[
F = \exp(\log R) \exp(\log U), \quad F^p = \exp(\log R^p) \exp(\log U^p)
\]

(47)
The exponentials \( \exp(\log U) \) and \( \exp(\log U^p) \) are computed from a spectral decomposition of \( \log U \) and \( \log U^p \), respectively. The exponential \( \exp(\log R) \) and \( \exp(\log R^p) \) are conveniently computed by Rodrigues’ formula. For completeness, these formulae are collected in Appendix A.

As mentioned in Section 2, the state variables are sampled at three quadrature points per element and extrapolated linearly to the domain of the element. In areas undergoing rapid transients, such as the nose of the projectile and the part of the target in contact with it, this piecewise linear interpolation may contain considerable noise away from the quadrature points. The deleterious effect of this noise can be mitigated by smoothing the state variable fields prior to effecting their transfer. We accomplish this smoothing by reducing the state data to the nodes so as to obtain a \( C^0 \) representation. The reduction to the nodes can be effected by recourse to the \( L^2 \)-projection

\[
A_s = \sum_b m_{ab} \sum_c \int_{\Omega^c} A^c N_b \, d\Omega
\]

(48)
where \( A_s \) is the nodal value of \( A \), \( A^c \) the local state variable field over element \( e \). \( \Omega^c \) is the domain of the element and

\[
m_{ab} = \sum_c \int_{\Omega^c} N_a N_b \, d\Omega
\]

(49)
Lumping the weights \( m_{ab} \), e.g. by row–sum, and computing all integrals by numerical quadrature yields the explicit formula

\[
A_s = \frac{1}{m_a} \sum_e \sum_q A^c_{q} N_a(\xi_q^e) w_q^c
\]

(50)
where \( \xi_q^e \) and \( w_q^c \), \( q = 1, 2, 3 \), are the quadrature points and weights for element \( e \) and \( A^c_q \) the corresponding values of the state variables. For the piecewise linear interpolation of the state variables employed here, Ortiz and Quigley’s [51] variationally consistent transfer operator, with all \( L^2 \) projections lumped, reduces simply to the evaluation of the old state variable representation at the quadrature points of the new mesh. The geometry of this operation applied to a smoothed state variable field is depicted in Fig. 5.

Finally, the treatment of rate-sensitivity requires special attention. The transferred stresses \( \tilde{\sigma} \) can be computed from \( F \) and \( F^p \) through the elastic relations (28). However, small variations in the corresponding effective stress

\[
\tilde{\sigma} = \sqrt{\frac{1}{2} \tilde{S}^{dev} \cdot \tilde{S}^{dev}}
\]

(51)
can lead to irreparably large spurious plastic strain rates when raised to a large exponent \( m \) as in the power law (31). To sidestep this difficulty, we scale the transferred deviatoric stresses \( \tilde{S}^{dev} \) so as to attain compatibility with the transferred plastic strain rates

\[
\tilde{\varepsilon}^p = \frac{\varepsilon^p - \varepsilon^p_{\infty}}{\Delta t}
\]

(52)
The correct scaling is

\[
\tilde{S}^{dev} \leftarrow \frac{\tilde{S}^{dev}}{\tilde{\sigma}} \left(1 + \frac{\varepsilon^p}{\varepsilon^p_{\infty}}\right)^{1/m}
\]

(53)
where, here and henceforth, the left arrow $\leftarrow$ is used to denote assignment. The complete stress tensor is synthesized as

$$\tilde{S} \leftarrow \tilde{S}^{\text{dev}} - pI$$

(54)

where $p$ is the transferred pressure field. Typically, this scaling introduces an exceedingly small—but critical—correction of the stress tensor. Equally small corrections of $F$ and $F^p$ need to follow to ensure compatibility with the elastic relations. This compatibility is achieved by inverting the elastic relations (28) for the elastic strains $(1/2) \log(C^e)$. The elastic stretch tensor $U^e$ follows by exponentiation of the elastic strains and the elastic deformation gradient is recovered from the polar decomposition

$$F^i \leftarrow R^e U^e$$

(55)

where $R^e$ is the rotation matrix corresponding to the transferred elastic deformation gradient $F^e$. In order to preserve plastic isochoricty, the slight variation in volume incurred as a result of (55) is absorbed to $F$, with the result

$$F_0 \leftarrow \left( \frac{J^e}{J} \right)^{1/3} F$$

(56)

where $J$ is computed from the transferred $F$ and $J^e$ from (55). The plastic deformation gradient finally follows as

$$F^p \leftarrow F^{e-1} F$$

(57)

which completes the rate-sensitivity correction. The complete sequence of steps followed for the transfer of state variable data is enumerated in Box 4.

We close this section by remarking on the possibility of appending equilibrium of the transferred stresses as an additional constraint on the transfer operator. Equilibrium can be enforced readily by a suitable $L^1$-projection of the transferred stresses. However, such a projection can be viewed as an additional step taken with $\Delta t = 0$. By causality, such a step can be lumped with the next time step with no loss of generality. This simply amounts to disregarding the equilibrium constraint and carrying the out-of-balance force on to the next time step.
Box 4
Transfer procedure

A. Set up auxiliary data structures
   (i) Quadtree of old corner nodes.
   (ii) Corner node attached element list.
B. Transfer nodal data.
   (i) Loop over new nodes.
   (ii) Find nearest old corner nodes by quadtree search.
   (iii) Search for attached element which contains new node.
   (iv) Interpolate nodal variables to new node.
C. Transfer element data.
   (i) Smooth state variables.
   (ii) Loop over new elements and their quadrature points.
   (iii) Find nearest old corner nodes by quadtree search.
   (iv) Search for attached element which contains new quadrature point.
   (v) Interpolate state variables to new quadrature point.
D. Enforce constraints on transferred data.

7. Benchmark test: Copper rod impact

The problem of a cylindrical rod which strikes a rigid surface head-on has been thoroughly
investigated in the past and, consequently, furnishes a convenient benchmark test [55]. The rod is made
of copper and initially has a radius of 3.2 mm and a length of 52.4 mm. The impact velocity is 227 m/s.
The deformation of the rod is presumed axisymmetric. For this simple geometry, the contact condition
can be modelled simply by constraining the velocities of the nodes in contact with the rigid surface.
The rod has a mass density $\rho = 8930 \text{ kg/m}^3$, a Young's modulus $E = 117 \text{ GPa}$, a Poisson's ratio $\nu = 0.35$,
and a yield stress $\sigma_y = 400 \text{ MPa}$. To facilitate comparisons with the results reported by Zhu and
Cescotto [55], the material is assumed to harden linearly with a plastic modulus $E_p = 100 \text{ MPa}$. All
calculations are carried out up to a time of 80 $\mu$s, at which point nearly all the initial kinetic energy has
been dissipated as plastic work [56]. All calculations are conducted on a DEC AXP 3000/80
workstation.

Three meshing strategies are compared: (a) no adaption; (b) adaption with partial coarsening; (c) adaption with full coarsening. The mesh size bounds are set to $h_{\text{min}} = 0.6 \text{ mm}$ and $h_{\text{max}} = 1.2 \text{ mm}$. In case (a), a uniformly fine mesh of size $h = h_{\text{min}}$ is used throughout the calculation. The initial mesh is identical in all three cases. Partial coarsening is obtained by inhibiting coarsening in elements with
effective plastic strains $\varepsilon_p > 0.1$. The deformed meshes at $t = 20 \mu$s, 40 $\mu$s and 80 $\mu$s are shown in Figs. 6–8. The deformation pattern is ostensibly similar in all cases. In solutions (b) and (c), a wave of mesh refinement emanates from the contact and propagates a distance along the bar. This refinement is followed by a wave of coarsening as the material unloads.

The distributions of equivalent plastic strain $\varepsilon_p$ at 80 $\mu$s resulting from the three meshing strategies are compared in Fig. 9. Solutions (a) and (b) are observed to be nearly identical. Solution (c) exhibits some jaggedness of the effective plastic strain contours due to excessive loss of spatial resolution. The computed final lengths of the bar, the final contact radii and maximum equivalent plastic strains are collected in Table 1, which also lists the results of Zhu and Cescotto [55] and DYNAD [57] for comparison. The final lengths and radii are ostensibly similar in all three cases, which is indicative of weak sensitivity of these parameters to the mesh adaption strategy. They also compare well with the results reported in [55] and [57], which validates our numerical procedure. As expected, the maximum effective plastic strains are obtained in case (a), with cases (b) and (c) behaving slightly more stiffly.

The superior performance of the adaptive strategy relative to fixed-mesh calculations is clearly evidenced by a comparison of time step histories, Fig. 10. Prior to 20 $\mu$s, the elements in the fixed-mesh calculation are highly distorted and the number of time steps required in all three cases is nearly identical. Beyond this point, however, the time step in the fixed-mesh calculation needs to be drastically
Fig. 6. Case (a): Deformed configurations without adaption.

Fig. 7. Case (b): Deformed configurations with adaption and partial coarsening.

reduced owing to severe element distortion near the contact. By way of contrast, adaptive meshing ensures good aspect ratios throughout the calculation, which in turn results in a nearly constant time step and a considerable reduction in the number of time steps. Thus, while 21 000 steps are required in the fixed-mesh calculation, 5000 steps suffice when mesh adaption is employed. An additional speed-up is obtained in consequence of the reduction in the number of elements afforded by mesh adaption, Fig. II. Thus, while the three solutions start out at 360 elements, rapid coarsening reduces the number of elements to a minimum in the range 220–230 in the adaptive solutions. The resulting CPU times: (a) 3567 s, (b) 601 s, and (c) 661 s, attest to a better than five-fold performance enhancement due mesh adaption in the particular benchmark test under consideration. The use of subcycling [58] can result in additional speed-up factors in the order of 2–3 in applications of the type envisioned here [20].
Fig. 8. Case (c): Deformed configurations with adaption and full coarsening.

Fig. 9. Comparison of plastic strain distributions.

Table I
Comparison of results for copper rod problem

<table>
<thead>
<tr>
<th>Case</th>
<th>Final length (mm)</th>
<th>Final mushroom radius (mm)</th>
<th>Max. equivalent plastic strain</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case (a)</td>
<td>21.42</td>
<td>7.24</td>
<td>3.25</td>
</tr>
<tr>
<td>Case (b)</td>
<td>21.44</td>
<td>7.21</td>
<td>3.01</td>
</tr>
<tr>
<td>Case (c)</td>
<td>21.44</td>
<td>7.21</td>
<td>2.97</td>
</tr>
<tr>
<td>Zhu &amp; Cescotto</td>
<td>21.26–21.40</td>
<td>6.97–7.18</td>
<td>2.75–3.03</td>
</tr>
<tr>
<td>DYN2A2D</td>
<td>21.47</td>
<td>7.13</td>
<td>3.05</td>
</tr>
</tbody>
</table>
§. Perforation of aluminum plates by conical-nosed projectiles

Forestal et al. [59] performed perforation experiments on 5003-11131 aluminum plates penetrated by tungsten heavy alloy (WHA) conical-nosed projectiles. These experiments provide a severe test of the predictive ability of the model. Plates 12.7 mm thick were tested at incident velocities in the range 220-115 m/s, and plates of 50.8 mm in thickness were tested in the range 50-1176 m/s. Thicker plates were also tested but are not considered here. All targets were square with a lateral dimension of 304 mm. We idealize the targets as equal area circular plates with a radius of 172 mm, and conduct all
simulations in the axisymmetric mode. The WHA projectiles weighed 25.6–25.9 g and remained essentially undeformed for incident velocities up to 1400 m/s. In simulations we have assumed a weight of 25.7 g for all projectiles.

Forrestal et al. [59] report stress-strain data for the 5083-H131 aluminum used in their experiments, which we have fitted to our power-law hardening model. The authors of Forrestal et al. [59] suggest that the friction between projectile and plate is minimal at best. Prompted by this observation we assume a vanishing friction coefficient. In addition, Forrestal et al. [59] obtained a good correlation between their analytical model and experimental data by assuming rate-independent behavior. However, at the exceedingly high rates of deformation of interest here a modulus of rate-sensitivity is to be expected (e.g. [61]) and, accordingly, we set \( m = 200 \). The Taylor–Quinney coefficient \( \beta \), which dictates the fraction of plastic work which is converted into heat, Eq. (16), is also set to zero. The remaining material parameters are collected in Tables 2 and 3.

The element size bounds for adaption are set to \( h_{\text{min}} = 2 \) mm and \( h_{\text{max}} = 10 \) mm. Complete coarsening is inhibited in regions where \( e^b > 0.01 \) when \( h \) grows to 5 mm. The adequacy of these parameters was tested by using a finer mesh in one of the simulations, which was found to have a negligible effect on residual velocities. The use of a pilot hole of 0.0001 mm surrounding the axis, such as advocated by Chen [60], was also found to have little or no effect on the results of the calculations. The historical effective plastic strain and mesh adaption in a 12.7 mm thick plate, \( V_r = 1195 \) m/s, and in a 50.8 mm thick plate, \( V_r = 1176 \) m/s, are shown in Figs. 12 and 13. As may be seen, a wave of mesh refinement accompanies the nose of the projectile, where the rate of deformation is particularly intense. The mesh is subsequently coarsened as the maximum perforation radius is attained and the material unloads. The mesh is partly coarsened only to prevent excessive loss of spatial resolution of the residual fields. As expected, the residual plastic deformation pattern corresponds to a nearly uniform radial expansion.

The history of the average velocity of the projectile is shown in Fig. 14 for 12.7 mm thick plates, and in Fig. 15 for 50.8 mm thick plates. The average velocity is computed so as to match the total linear momentum of the projectile, which leads to the expression

\[
V = \frac{\sum_{a=1}^{N_p} m_a V_a}{\sum_{a=1}^{N_p} m_a}
\]

(50)

where \( m_a \) and \( V_a \) are the mass and axial velocity of node \( a \), respectively, and the sum extends to the nodes in the projectile. The deceleration of the projectile increases gradually as the conical nose penetrates the target, attains a nearly constant value when the full length of the projectile is contained in the plate, and begins to decrease as the projectile nose breaks through the rear surface of the plate. Eventually, the velocity attains a constant residual value as the projectile exits the plate.

Table 2
Mechanical material constants

<table>
<thead>
<tr>
<th>Material</th>
<th>( \rho ) (kg/m(^3))</th>
<th>( E ) (GPa)</th>
<th>( \nu )</th>
<th>( \sigma_y ) (MPa)</th>
<th>( \varepsilon_{pl}^* )</th>
<th>( m )</th>
<th>( n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>5083-H131 Aluminum</td>
<td>2660</td>
<td>70.3</td>
<td>0.33</td>
<td>276</td>
<td>0.003926</td>
<td>200</td>
<td>1.3</td>
</tr>
<tr>
<td>Tungsten heavy alloy</td>
<td>18 500</td>
<td>345</td>
<td>0.29</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>—</td>
</tr>
</tbody>
</table>

Table 3
Equation of state constants

<table>
<thead>
<tr>
<th>Material</th>
<th>( \Gamma )</th>
<th>( K_1 ) (GPa)</th>
<th>( K_2 ) (GPa)</th>
<th>( K_3 ) (GPa)</th>
</tr>
</thead>
<tbody>
<tr>
<td>5083-H131 Aluminum</td>
<td>2.0</td>
<td>70.6</td>
<td>128.3</td>
<td>125.1</td>
</tr>
<tr>
<td>Tungsten heavy alloy</td>
<td>1.43</td>
<td>302.1</td>
<td>469.8</td>
<td>334.9</td>
</tr>
</tbody>
</table>
Fig. 12. Perforation of 12.7 mm thick aluminum plate, \( V_i = 1195 \) m/s.

Fig. 16 shows the computed variation of the residual velocity \( V_r \) with the incident velocity \( V_i \). The experimental data of Forrestal et al. [59] is also shown for comparison. The agreement between simulation and experiment is excellent, which provides a first indication as to the adequacy of the assumed material parameters. In order to ascertain the effect of rate-dependency, we have repeated the simulations with \( m = \infty \), corresponding to rate-independent behavior. Due to the diminished viscous resistance to penetration, the computed residual velocities now overestimate the experimental data by as much as 20 m/s, Fig. 17.

We have also tested the adequacy of the assumptions of frictionless contact between projectile and plate, and of no plastic work converted to heat. When the friction coefficient is nonzero—and values suited to aluminum are assigned to the thermal capacity and conductivity of the plate—our simulations reveal the formation of an exceedingly narrow boundary layer at the contact between projectile and plate. The temperatures within the layer reach the melting temperature \( T_m \) and, consequently, the flow stress vanishes. The net effect of this mechanism is to defeat friction by lubricating the contact. The thinness of the boundary layer further justifies its idealization as a frictionless surface.

Simulations with a Taylor-Quinney coefficient \( \beta = 0.9 \) were also conducted with a view to gaining insight into the role of this parameter. For the thin plates, the residual velocities remain nearly unchanged at high incident velocities, but are overpredicted by as much as 40 m/s at low incident velocities. For the thick plates, the residual velocities are overpredicted by up to 50 m/s and 30 m/s at low and high incident velocities, respectively. These results suggest that the strength of the material plays a significant role at low incident velocities but is overshadowed by inertia at high incident velocities.
Fig. 13. Perforation of 50.8 mm thick aluminum plate, \( V = 1176 \text{ m/s} \).

Fig. 14. Average velocity history, plate thickness = 12.7 mm.
velocities, in keeping with intuition. In consequence, the softening effect of thermal coupling is more acute for slow projectiles. The consistent overprediction of the residual velocities in this range which results from consideration of thermal coupling leads us to conjecture that, indeed, little plastic work is converted into heat in aluminum deformed at very high strain rates and temperatures. Interestingly, recent experiments by Rosakis and Ravichandran [62] show that $\beta$ is a decreasing function of strain rate in the range $7 \times 10^3 - 4 \times 10^4 \, \text{s}^{-1}$, which lends some experimental support to our conjecture.
9. Penetration of steel targets by WHA long rods

Anderson et al. [63] investigated the ballistic behavior of high-strength steel armor penetrated by tungsten heavy alloy (WHA) long-rod projectiles. Plates 29.0 and 49.5 mm thick were tested at incident velocities of 1250 and 1700 m/s, respectively. These thicknesses represent previously determined ballistic limits for the test impact velocities. The projectile had the form of a blunt cylinder 2 mm in radius and 50 mm in length. The lateral dimensions of the rectangular steel plate were 40 mm by 70 mm. Flash X-ray images of the penetration process were taken at regular intervals of time, yielding detailed nose and tail trajectories. In addition, Anderson et al. [63] simulated the tests using the Eulerian hydrocode CTH and the Johnson–Cook model [64], and obtained good agreement with the experimental data.

As in the preceding section, we idealize the target as an equal area circular plate of 30 mm in radius. We have calibrated our constitutive relations so as to match the material data used by Anderson et al. [63] in their simulations. The resulting material constants are listed in Tables 4–6. The Taylor–Quinney coefficient $\beta$ is set to 0.9. The element size bounds are $h_{\text{min}} = 0.5$ mm, $h_{\text{max}} = 2$ mm, and the

<table>
<thead>
<tr>
<th>Material</th>
<th>$\rho$ (kg/m$^3$)</th>
<th>$E$ (GPa)</th>
<th>$\nu$</th>
<th>$\sigma_0$ (GPa)</th>
<th>$\sigma_0^*$</th>
<th>$m$</th>
<th>$\beta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>High-strength steel</td>
<td>7850</td>
<td>200</td>
<td>0.29</td>
<td>1.50</td>
<td>0.001</td>
<td>340</td>
<td>2</td>
</tr>
<tr>
<td>Tungsten heavy alloy</td>
<td>17600</td>
<td>323</td>
<td>0.30</td>
<td>1.51</td>
<td>0.0001</td>
<td>68</td>
<td>9</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Material</th>
<th>$m$</th>
<th>$K_0$ (GPa)</th>
<th>$K_1$ (GPa)</th>
<th>$K_2$ (GPa)</th>
</tr>
</thead>
<tbody>
<tr>
<td>High-strength steel</td>
<td>1.16</td>
<td>163.9</td>
<td>294.4</td>
<td>500.0</td>
</tr>
<tr>
<td>Tungsten heavy alloy</td>
<td>1.43</td>
<td>302.1</td>
<td>469.8</td>
<td>334.9</td>
</tr>
</tbody>
</table>
Table 6
Thermal constants

<table>
<thead>
<tr>
<th>Material</th>
<th>( c ) (W/m·K)</th>
<th>( k ) (W/kg·K)</th>
<th>( L_s ) (K)</th>
<th>( L_a ) (K)</th>
<th>( \theta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>High-strength steel</td>
<td>477</td>
<td>38</td>
<td>294</td>
<td>1777</td>
<td>1.17</td>
</tr>
<tr>
<td>Tungsten heavy alloy</td>
<td>134</td>
<td>75</td>
<td>294</td>
<td>1723</td>
<td>1.0</td>
</tr>
</tbody>
</table>

limit is \( h = 1 \) mm in regions where \( e^p \geq 0.10 \). A relatively fine mesh is required to model the erosion at the nose of the projectile, the projectile ejecta, and the projectile/target interface. The friction coefficient \( \mu \) between projectile and plate is set to 0.20. As in the case of aluminum armor, however, friction is defeated by thermal softening at the projectile/plate interface and its effect on the solution is almost entirely lost.

The deformed configurations and distributions of effective plastic strain at times \( t = 20 \) \( \mu \)s, 40 \( \mu \)s, 60 \( \mu \)s and 80 \( \mu \)s are shown in Figs. 18 and 19. The final configurations are composed of 2109 nodes and 87 elements for \( V_i = 1250 \) m/s, and of 3637 nodes and 1687 elements for \( V_i = 1700 \) m/s. As the projectile penetrates the target it mushroom and lines the crater. Portions of the ejecta come in contact with the shank of the projectile. Eventually, the projectile is reduced to a stub and penetration ceases, at which point the projectile recoils slightly.

As expected, the primary active region is the neighborhood of the tip of the projectile, where maximum refinement is attained. This corresponds to the 'process zone' described by Mescall [65] for penetration by an eroding projectile. Away from this zone, the shank remains essentially elastic and travels at near the incident velocity. The target primarily undergoes radial expansion, as evidenced by the level contours of plastic strain. The crater diameters at the plate surface are computed to be 7.3 mm for \( V_i = 1250 \) m/s, and 8.0 mm for \( V_i = 1700 \) m/s, which are in fair agreement with the experimentally measured values of 7.4 mm and 8.3 mm, respectively. Contours of temperature 20 \( \mu \)s after impact are shown in Fig. 20 for an incident velocity \( V_i = 1700 \) m/s. Peak temperatures in the order of 1700 \( ^\circ \)K arise at the projectile/target interface. These temperatures are close to the melting point of both steel and the WHA. The sharp temperature gradients which develop near the projectile/target interface are also noteworthy.

It is interesting to note that the projectile ejecta protrude out of the target crater in the \( V_i = 1250 \) m/s case, but remain contained within the crater in the \( V_i = 1700 \) m/s case. This difference in behavior can be rationalized as follows. The plastic incompressibility of the projectile yields the identity

\[ V_e = \left( A_p + A_e \right) U - A_p V_i \]

where \( U, V_i \) and \( V_e \) are the projectile/target interface, incident and ejectum velocities, respectively, with the velocity sign taken as positive downwards, and \( A_p \) and \( A_e \) are the cross-sectional areas of the projectile and the ejectum, respectively. From the interface or projectile-nose velocity plots in Figs. 23 and 24, the average interface velocities are estimated as \( U_{ave} = 450 \) m/s for \( V_i = 1250 \) m/s, and \( U_{ave} = 800 \) m/s for \( V_i = 1700 \) m/s. The calculated average ejectum area is \( 1.55 \times 10^{-3} \) m² for \( V_i = 1250 \) m/s, and \( 1.38 \times 10^{-5} \) m² for \( V_i = 1700 \) m/s. Insertion of these data into (59) gives net ejecta velocities of \(-200 \) m/s (upward) for \( V_i = 1250 \) m/s and 22 m/s (downward) for \( V_i = 1700 \) m/s, which explains the aforementioned difference in behavior.

The computed nose and tail trajectories are compared with the data of Anderson et al. [63] in Figs. 21 and 22. The nose and tail are identified with the projectile boundary nodes lying on the axis. As may be seen, the agreement between simulation and experiment is excellent. Our computed nose and tail trajectories also compare well with the CTH results of Anderson et al. [63]. The computed and observed nose and tail velocity histories are shown in Figs. 23 and 24. The experimental velocities are computed by fitting the trajectories smoothly and differentiating with respect to time. Evidently, only average trends can be inferred by this means. The experimental tail velocity history is replicated closely by the calculations. The tail speed remains near the incident velocity for about 50 \( \mu \)s and then drops.
Fig. 15. Deformed configurations and plastic strain contour plots, plate thickness = 20 mm, $V_0 = 1250$ m/s.
Fig. 19. Deformed configurations and plastic strain contour plots. Plate thickness = 49.5 mm. $V = 1700$ m/s.
Fig. 20. Mesh and temperature contour plots. plate thickness = 49.5 mm, $V_i = 1700$ m/s.

Fig. 21. Projectile nose and tail trajectories plate thickness = 70 mm, $V_i = 1760$ m/s.
Fig. 22. Projectile nose and tail trajectories, plate thickness = 49.5 mm, \( V_i = 1700 \text{ m/s} \)

Fig. 23. Projectile nose and tail velocity histories, plate thickness = 29 mm, \( V_i = 1250 \text{ m/s} \).

rapily to zero. In the average sense permitted by the sparciness of the data, the nose velocity history is also well matched by the calculations. However, in this case the numerical record exhibits rapid oscillations caused by the high-frequency response of the elements in the process zone.

9. Summary and discussion

We have developed a computational capability which enables the simulation of metal target penetration by deformable projectiles. The field equations are modelled within a Lagrangian framework, which facilitates the treatment of inelasticity, traction-free boundary conditions, contact between
deformable surfaces, as well as the computation of residual stresses following unloading. Adaptive meshing plays a key role in alleviating element distortions and modelling accurate regions of high variation in the velocity field. Continuous remeshing provides an alternative to the common practice of removing failed elements in Lagrangian codes. The main building blocks of the adaptive meshing scheme are an automatic mesh generator, mesh refinement/coarsening strategies and a transfer operator. The model uses a robust contact/friction algorithm capable of handling self-contact and multi-body contact. A staggered procedure is used to take thermomechanical coupling into account. Material response is modelled through an equation of state and a thermoviscoplastic formulation. The predictive ability of the method has been demonstrated in three applications, where close correlation with experimental and numerical results has been achieved.

The simulations of aluminum armor penetration reveal intriguing insights into material behavior in regimes which are poorly understood at present, whether from a theoretical or an experimental perspective. In particular, we find that best agreement with experiment is obtained by taking the Taylor–Quinney coefficient $\beta$ to vanish. This is tantamount to assuming that the fraction of plastic work which is converted into heat is negligible under the extreme conditions of strain rate and temperature which prevail in ballistic penetration. Recent experiments of Rosakis and Ravichandran [62] show that $\beta$ is, indeed, a decreasing function of strain rate in the range of $2 \times 10^3$–$4 \times 10^3$ s$^{-1}$. Whether this trend continues up to higher strain rates does not appear to be known at present. A plausible micromechanical scenario which explains these observations is the following. At low strain rates, dislocations in aluminum organize into low-energy structures. Under these conditions, the stored energy amounts to a small fraction of the work of deformation, most of which is dissipated as heat. However, at very high strain rates the dislocations may not have time to attain low-energy configurations and, consequently, store away the greater part of the work of deformation, with little or no heat production. The self-organization of the dislocations may also be hindered by entropic effects at high temperatures. Further experimental and theoretical work is clearly needed in order to develop a full understanding of this phenomenon.

In the ballistic penetration applications pursued here, radial expansion of the plate and erosion of the projectile are the dominant deformation mechanisms. When the ballistic limit of the armor is exceeded, other penetration mechanisms, such as shear banding and plugging, may become operative. Fracture can also play an important role, especially where brittle armor is concerned. Marusich and Ortiz [30]
have developed an adaptive meshing method for initiating and propagating ductile and brittle multiple cracks along arbitrary paths. The approach has proven effective at predicting chip morphologies in high-speed machining simulations. Camacho [15] and Camacho and Ortiz [17,66] have also demonstrated how processes of fragmentation in brittle solids can be followed explicitly by recourse to automatic meshing, cohesive elements, and multi-body contact. These developments open avenues for extending the analyses presented in this paper so as to account for material failure.

Acknowledgments

The work has been funded by the Army Research Office University Research Initiative through grant DAAL 03-92-G-0107. The authors are indebted to Prof. R. Clifton, Dr. J. Nagtegaal of HKS Inc. and Dr. M. Raffenberg of BRL for helpful suggestions and discussions.

Appendix A. Exponential and logarithmic formulae

Let \( R \) be a proper orthogonal \( 3 \times 3 \) matrix. The unit of rotation \( p \) is explicitly given by the formulae [54]

\[
p_1 = \frac{R_{12} - R_{21}}{2 \sin \theta}, \quad p_2 = \frac{R_{13} - R_{31}}{2 \sin \theta}, \quad p_3 = \frac{R_{23} - R_{32}}{2 \sin \theta},
\]

(A.1)

where the angle of rotation \( \theta \) can be computed from the identity

\[
I_3 = \text{trace}(R) = 1 + 2 \cos \theta
\]

(A.2)

The axial vector of \( R \) is

\[
w = 6p
\]

(A.3)

where \( \log R \) follows as

\[
\log R = \begin{pmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{pmatrix} = W
\]

(A.4)

The matrix \( R \) can be recovered from \( W \) by recourse to Rodrigues’ formula

\[
R = \exp(W) = \cos|w|I + \sin|w| \frac{W}{|w|} + (1 - \cos|w|) \frac{w \otimes w}{|w|^2}
\]

(A.5)

References


