\[
\cos \varphi \frac{d\varphi}{\varphi}, \quad (A.4)
\]
\[
(A.5)
\]
\[
21.1, 9.121.7 \text{ and } J, \quad (A.6)
\]
\[
\frac{dz}{\sqrt{\varphi}} \quad (A.7)
\]
\[
\frac{1}{2} \ln(1 - \omega^2)
\]
\[
\text{constant.} \quad (A.8)
\]

THE TWO-DIMENSIONAL STRUCTURE OF DYNAMIC BOUNDARY LAYERS AND SHEAR BANDS IN THERMOVISCOPlastic SOLIDS

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ABSTRACT

A general boundary layer theory for thermoviscoplastic solids which accounts for inertia, rate sensitivity, hardening, thermal coupling, heat convection and conduction, and thermal softening is developed. In many applications of interest, the boundary layer equations can be considerably simplified by recourse to similarity methods, which facilitates the determination of steady-state and transient fully non-linear two-dimensional solutions. A simple analysis of the asymptotic behavior of the steady-state solutions leads to a classification of stable and unstable regimes. Under adiabatic conditions, the resulting material stability criterion coincides with that previously derived by Molinari and Clifton ([1987] Analytical characterization of shear localization in thermoviscoplastic solids. J. Appl. Mech. 54, 806–812) by a quasi-static, one-dimensional analysis. The transition from initially stable to unstable behavior can also be conveniently described by similarity methods. This provides a powerful semi-analytical tool for the interpretation of impact tests exhibiting dynamic shear bands, and for the characterization of the two-dimensional structure of such bands. It follows from the theory that, if the velocity of the impactor is held steady, the leading tip of the shear band propagates at a constant speed. This shear band speed follows readily from the theory as a function of the impact velocity and material parameters. The two-dimensional velocity, stress, temperature and plastic work fields attendant to the propagating shear band are also determined.

1. INTRODUCTION

Solids deforming at high rates often develop narrow layers of intense shearing. Areas of application where such layers arise include ultra-high speed machining (Komanduri et al., 1984), geologic impact cratering (Melosh, 1989; O'Keefe and Ahrens, 1993), and terminal ballistics (Backman, 1969; Craig and Stock, 1970; Brunton et al., 1964; Stock and Thompson, 1970; Backman and Finnegan, 1973; Wingrove, 1973; Rogers, 1979). Outstanding features of dynamic shear bands are their thinness, with typical widths of 10–100 µm (Leech, 1985); high local shear strains, which can reach values of up to 100 (Wingrove, 1973; Rogers, 1979; Timothy, 1987); ultra-high local shear strain rates, often in excess of \(10^6 \text{s}^{-1}\) (Wingrove, 1973; Rogers, 1979; Timothy, 1987); local temperature rises of several hundred degrees (Costin et al., 1979; Hartley, 1986; Hartley et al., 1987; Marchand and Duffy, 1988; Zhou et al., 1995); and high propagation speeds, sometimes in excess of 1000 m s\(^{-1}\) (Mason et al., 1994; Zhou et

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al., 1995). In addition, cracks—whether the result of brittle fracture (Backman and Finnegian, 1973; Dormeaval, 1987), or of microvoid growth and coalescence (Johnson et al., 1983; Huang, 1987; Giovannola, 1988; Xu et al., 1989; Mason et al., 1994; Zhou et al., 1995)—often form along shear bands.

The realistic modeling of these problems requires consideration of large plastic deformations, rate sensitivity, hardening, heat convection and conduction, thermal softening and inertia effects (Backman and Goldsmith, 1978; Wilkins, 1978; Zukas, 1990). Fully non-linear multidimensional solutions to problems of this nature are rare (see Wright and Walter, 1994, for a notable exception). However, the thinness of the shear layers of interest here makes possible certain approximations in the governing equations which facilitate the analytical characterization of the flow. The systematic use of these approximations results in a much simplified set of boundary layer equations which, in some cases, lend themselves to analytical treatment.

Boundary layer theory is a well-developed discipline as it relates to fluid flows. In particular, the boundary layers of Newtonian and non-Newtonian fluids have been extensively investigated, including thermal effects (Rosenhead, 1963). However, solids differ from fluids in several notable respects, even when extensive plastic flow is involved. Thus, solids harden, or sometimes soften with plastic deformation, which has a marked influence on the nature of the flow. In particular, strong softening can give rise to material instabilities resulting in plastic flow localization. Also significant is the fact that the ranges of common physical properties, such as viscosity, mass density, heat conduction, heat capacity, and others, differ widely from solids to fluids, which in turn gives rise to wide disparities in behavior.

In this paper, we formulate a general boundary layer theory for thermoviscoplastic solids which accounts for inertia, rate sensitivity, hardening, thermal coupling, heat convection and conduction, and thermal softening. In accordance with the flow character of the solutions sought, we adopt an eulerian description of the motion. The principal simplifying assumption underlying the theory is that the solution is rapidly varying across the thickness of the boundary layer, but slowly varying in the remaining directions. Thus, if the solution varies appreciably over lengths of order $L$ along directions which are parallel to the layer, then it is presumed to vary appreciably over lengths of order $\varepsilon L$ across the layer, where $\varepsilon$ is a small parameter. The reduced boundary layer equations are obtained by this scaling argument in Section 2.2. The boundary layer equations of linear momentum balance, energy balance and hardening follow asymptotically to leading order in $\varepsilon$. The scaling argument also provides a physical interpretation for the parameter $\varepsilon$, which is found to be a power of a generalized Reynolds number of the flow. Consequently, the asymptotic regime in which boundary layer theory applies is characterized, as expected, by high Reynolds numbers.

We apply the boundary layer theory to the determination of the two-dimensional structure of dynamic shear bands in thermoviscoplastic solids. For definiteness, we specifically consider the case of a plate which is impacted upon by a flat-ended, rigid projectile. When the impact velocity is sufficiently high, a sharp shear band is often observed to propagate deep into the plate from the edge of the impactor. For instance, Wingrove (1973) studied the penetration of 2014-T6 aluminum alloy plates by flat-ended projectiles. Intense shear bands were observed to be punched through the thickness of the target sheet. Shear bands of a different type were formed by Kalthoff (1970) at impact and in notched plates dynamically loaded with a blunt notch (Kalthoff, 1995).

While these tests are physically different, the results can be obtained from an analysis of the flows of a fluid that is maintained over a boundary. We assume that the boundary layer governs. Although the boundary layer theory is usually applied to dynamic flows even in the newtonian regime, it may be especially appropriate for the analysis of the reduced governing equations. The problem is one of a boundary layer, even in the newtonian regime, and plastic work is often poorly approximated by an ideal boundary layer model, even with the best of approximations. The singularity structure of the governing equations is not captured in the usual boundary layer formulation, and the boundary layer is often not a useful tool in the analysis of the problem. The boundary layer theory presented here attests to the need for a general stability analysis for this type of problem. The analysis has been developed for the case of a plate with no pre-existing yield, but the method is general and may be applied to cases with pre-existing yield. The method is based on a partial differential equation for the displacement, and is applicable to a wide range of materials. The solution obtained is of the form $\varepsilon^2$ in the plastic region, and $\varepsilon$ in the near-plastic region. The solution provides a good approximation to the actual displacement in the plastic region, and the near-plastic region is characterized by a sharp transition. The solution is valid for arbitrary values of $\varepsilon$, and is particularly useful for very thin plates. The method is also applicable to the case of a pre-existing yield, and may be extended to the case of a plate with a pre-existing crack. The method is also applicable to the case of a plate with a pre-existing hole, and may be extended to the case of a plate with a pre-existing notch. The method is also applicable to the case of a pre-existing yield, and may be extended to the case of a plate with a pre-existing crack. The method is also applicable to the case of a pre-existing hole, and may be extended to the case of a plate with a pre-existing notch. The method is also applicable to the case of a pre-existing yield, and may be extended to the case of a plate with a pre-existing crack. The method is also applicable to the case of a pre-existing hole, and may be extended to the case of a plate with a pre-existing notch. The method is also applicable to the case of a pre-existing yield, and may be extended to the case of a plate with a pre-existing crack. The method is also applicable to the case of a pre-existing hole, and may be extended to the case of a plate with a pre-existing notch.
Dynamic boundary layers and shear bands

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...of large plastic duction, thermal ns, 1978; Zukas, is nature are rare ne thinness of the in the governing r. The systematic uary layer equa-
to fluid flows. In fluids have been . However, solids e plastic flow is formation, which ng softening can Also significant s viscosity, mass m solids to fluids,...

...ermoviscoplastic coupling, heat ce with the flow n of the motion. it the solution is vly varying in the shths of order L vary appreciably ter. The reduced Section 2.2. The ce and hardening t also provides a be a power of a ptotic regime in high Reynolds two-dimensional r definiteness, we flat-ended, rigid ear band is often tor. For instance, oy plates by flathed through the...
band tip speed is found to be greatly in excess of the impact velocity, in agreement
with the observations of Zhou et al. (1995). The ratio of the tip speed to the impact
velocity rises steeply as a function of the latter at low impact velocities, also in keeping
with the observations of Zhou et al. (1995), and saturates at high impact velocities.
The tip velocity depends sensitively on material parameters such as rate sensitivity,
rate of hardening and rate of thermal softening. These dependencies, as well as the
two-dimensional transient velocity, stress, temperature and plastic work fields follow
from the theory and are reported in Section 5.3.

2. BOUNDARY LAYER EQUATIONS

Under certain conditions, the plastic flow of solids may be expected to be confined
to layers which are thin relative to all other geometrical dimensions of the problem.
The thinness of the layer makes possible certain approximations in the governing
equations which facilitate the characterization of the flow within the layer. For
fluids, the hypothesis that viscosity effects are significant only in narrow layers, the
thicknesses of which approach zero as the Reynolds number increases to infinity, was
advanced by Prandtl (1904), who also proceeded to compute the simplified boundary
layer equations of motion.

In this section, we formulate similar approximations for thermoviscoplastic solids.
We follow the original method advanced by Prandtl (1904) and Blasius (1908) based
on a consideration of approximate orders of magnitude. In applications, we shall
confine our attention to straight layers. Extensions to flows along curved surfaces and
to flows past bodies of revolution entail the addition to the boundary layer equations
of terms which are geometrical in nature. Detailed accounts of these extensions may
be found in standards texts (see, e.g. Rosenhead, 1963) and will not be pursued here.

2.1. Governing equations

We start by writing out the general equations of motion of the solid. In accordance
with the flow character of the solutions to be sought, we adopt an eulerian description
of the motion. Let \( \rho \) be the mass density, \( \mathbf{v} \) the eulerian velocity field, \( s_{ij} \) the stress
deviator tensor, and \( p \) the hydrostatic pressure, which we shall take to be positive in
tension. The equation of linear momentum balance is

\[
\rho (v_i + v_j v_{ij}) = s_{ij,j} + p_{,j},
\]

where commas are used to denote partial differentiation. We shall assume the elastic
strains to be negligible compared to plastic deformations, and the plastic flow to be
volume preserving, so that

\( v_{ij} = 0. \)

We further postulate the existence of a plastic flow potential \( D(d, \theta, \mathbf{q}) \) such that

\[
s_{ij} = \frac{\partial D}{\partial d_{ij}},
\]

where \( d_{ij} = (v_{ij} + i) \) and \( \mathbf{q} \) is some suitable reference stress and we
confine our attention to incompressible materials.

In addition, we make a further approximation by neglecting the hardening of the
material.

For a solid obeying

\[
\frac{\partial w}{\partial t} + \nabla \cdot \mathbf{q} = 0,
\]

where

\[
is an effective stress and the evolution of \( w \) is,
\]

The temperature

\[
\frac{\partial T}{\partial t} + \nabla \cdot \theta = 0,
\]

where \( c \) is the heat capacity coefficient, whilst

\[
E \text{ we adopt the form}
\]

where \( g_t = \theta_j \) is the

\[ \dot{\tau} = (2d_i/d_{ij})^{1/2}. \] (4)

In addition, we choose to identify the sole state variable describing the state of hardening of the solid with the plastic work

\[ w = \int_0^t \tau \dot{\tau} dt. \] (5)

For a solid obeying \( J_2 \)-flow theory, (5) reduces to

\[ w = \int_0^t \tau \dot{\tau} dt, \] (6)

where

\[ \tau = (s_{ij} s_{ij}/2)^{1/2} \] (7)

is an effective shear stress. From (5), it follows that the equation governing the evolution of \( w \) is, simply,

\[ w_{,i} + v_i w_{,i} = \dot{w} = s_{ij} d_{ij}. \] (8)

The temperature distribution is governed by the energy balance equation

\[ \rho c (\dot{\theta} + v_i \dot{\theta}_i) = q_{,i} + \beta s_{ij} d_{ij}, \] (9)

where \( \rho \) is the heat capacity, \( q_i \) is the heat flux tensor, and \( \beta \) the Taylor–Quinney coefficient, which represents the fraction of plastic work converted into heat. The system of equations is completed by assuming a generalized Fourier law of the form

\[ q_i = \frac{\partial E}{\partial \theta_i}, \] (10)

where \( g_i = \theta_{,i} \) is the temperature gradient and \( E \) is a function of \( g_i \) and \( \theta \). For \( c, D \) and \( E \) we adopt the following power-law expressions:

\[ c = c_0 \left( \frac{\theta}{\theta_0} \right)^q, \] (11)

\[ D = \frac{\sigma_0 \dot{\gamma}_0}{m+1} \left( \frac{\dot{\gamma}}{\dot{\gamma}_0} \right)^{m+1} \left( \frac{w}{w_0} \right)^a \left( \frac{\theta}{\theta_0} \right)^t, \] (12)
\[ E = \frac{q_0 \gamma_0}{k+1} \left( \frac{\theta}{\theta_0} \right)^{\gamma} \left( \frac{g}{g_0} \right)^{k+1}, \]  

where we have denoted

\[ \bar{n} = \frac{n}{n+1}. \]  

In (11), \( c_0 \) and \( \theta_0 \) are reference values of the heat capacity and temperature, respectively, and \( q \) is an exponent characteristic of the variation of \( \nu \) with temperature over the range of interest. In (12), \( \sigma_0 \) is the flow stress, \( \theta_0 \), \( w_0 \) and \( \gamma_0 \) are reference values of temperature, plastic work and strain rate, respectively, \( l \), \( n \) and \( m \) are the thermal softening, strain hardening and rate sensitivity exponents, respectively, and we have written \( g = |g| \). In (13), \( q_0 \) is a characteristic heat flux, \( \theta_0 \) and \( g_0 \) are reference values of temperature and temperature gradient, respectively, \( p \) is an exponent characteristic of the variation of \( G \) with temperature over the range of interest, and \( k \) a thermal conductivity exponent.

It is sometimes convenient to collect all multiplicative constants in (11)–(13), and to recast \( c \), \( D \) and \( E \) in the simpler form

\[ c = C \theta^n, \quad D = A \frac{m+1}{m+1} \gamma^{m+1} \theta^{n} w^n, \quad E = B \frac{k+1}{k+1} g^{k+1} \theta^p, \]  

with

\[ C = c_0 \theta_0^{-q}, \quad A = \sigma_0 \gamma_0^{-m} w_0^{-n} \theta_0^{-l}, \quad B = q_0 \theta_0^{-p} g_0^{-k}. \]  

In the Newtonian case, which is recovered by the choice of constants \( m = 1, n = l = 0 \), \( A \) reduces to the viscosity of the medium. In the special case of \( p = 0, k = 1 \), which corresponds to Fourier’s law of heat conduction, \( B \) defines the thermal conductivity of the solid.

Critical aspects of the theory, such as the scaling argument upon which the derivation of the boundary layer equations is predicated, and the ability to obtain similarity solutions, rest upon the assumption of power-law behavior expressed in (11)–(13). Conveniendy, power-law expressions are often used to fit experimental data. Typical values of the material constants are given in Table 1, in which we have set \( w_0 = \sigma_0 \gamma_0 \). The mechanical parameters are taken from Klopp et al. (1985) and Tong et al. (1992), whereas the heat transport parameters are either taken directly from Bayley et al. (1972), or computed from data presented therein. No data could be found on the thermal conductivity exponent \( k \), and therefore we simply set it equal to 1. The constant \( K_0 \equiv q_0 / g_0 \) then defines the thermal conductivity of the solid at the reference temperature \( \theta_0 \).

The deformation processes of interest here involve high rates of shearing, often in excess of \( 10^6 \) s\(^{-1} \). Therefore, the values of the rate sensitivity exponent \( m \) listed in Table 1 are representative of material behavior in this range. In particular, the appropriate values of \( m \) are considerably higher than those corresponding to low or moderate strain rates. The thermal sensitivity exponent \( l \) is negative for most materials, which implies softening of the solid with increasing temperature. Positive values of \( l \), implying thermal hardening, are observed (Eleich et al., 1983), but must be carefully noted.

2.2. Boundary layer flows

Next we proceed to consider boundary layer flows. As in Pr \( 10^3 \) rapidly varying in several directions, we consider a plate of the same kind. Let it be the orthogon to the other. If we further forego the thin layer of solid on the surface. The equation moviscoplastic mixed mode.

We begin by considering the stress field on the surface of the plate. We set a small param...
(13)

perature, respec-
temperature over
ference values of
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cely, and we have
reference values
t characteristic
and \( k \) a thermal
in (11)–(13), and

(14)

implying thermal hardening, are also observed in some intermetallic compounds, but
this behavior is considered anomalous. The reference strains \( \gamma_0 \) in Table 1 should also
be carefully noted. These relatively large deformations are commensurate with typical
critical strains for localization (Bai and Dodd, 1992). In this range, the rate of
hardening is small and the rate of thermal softening comparatively high. In the early
stages of deformation, higher values of \( n \) and less negative values of \( l \) are often
observed (Eleiche and Duffy, 1975; Eleiche and Campbell, 1976), which adds to the
stability of the material.

2.2. Boundary layer equations

Next we proceed to simplify the general equations of motion for a special class of
flows. As in Prandtl's original work, we seek to characterize solutions which are
rapidly varying in one direction, while being slowly varying in the remaining ortho-
gonal directions. As an illustration of how such deformation fields may arise in solids,
consider a plate impacted on one side by a flat-ended impactor traveling at speed \( U \),
Fig. 1(a). Let the \( x_1 \) axis point into the plate from the edge of the impactor, and \( x_2 \)
be the orthogonal direction within the plane of the plate. Imagine cutting the plate
along the \( x_1 \) axis so that its top and bottom parts can slide freely relative to each
other. If we further idealize the material as rigid–plastic in the manner outlined in the
foregoing, the impacted portion of the plate will move rigidly with velocity \( U \), Fig.
1(b). Evidently, the resulting velocity field is incompatible along the \( x_1 \) axis. If \( U \) is
sufficiently large, we expect compatibility to be restored through the development of
a thin layer of shearing deformation, or boundary layer, Fig. 1(c).

The equations which determine the structure of boundary layers in thermoviscoplastic solids can be obtained by recourse to the following scaling argument.

We begin by assuming that the solution varies appreciably in the \( x_1 \) direction over
lengths of order \( L \), and over the \( x_2 \) direction over lengths of order \( \varepsilon L \), where \( \varepsilon \ll 1 \) is
a small parameter. Next we proceed to size the various terms in the governing

<table>
<thead>
<tr>
<th>Constant</th>
<th>Pure Al</th>
<th>Iron</th>
<th>OFHC Cu</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sigma_0 ) (MPa)</td>
<td>125</td>
<td>420</td>
<td>265</td>
</tr>
<tr>
<td>( \theta_0 ) (K)</td>
<td>295</td>
<td>295</td>
<td>295</td>
</tr>
<tr>
<td>( \gamma_0 )</td>
<td>0.05</td>
<td>0.15</td>
<td>0.2</td>
</tr>
<tr>
<td>( \gamma_0 ) (s(^{-1}))</td>
<td>1.53(10^5)</td>
<td>4(10^5)</td>
<td>5(10^5)</td>
</tr>
<tr>
<td>( m )</td>
<td>0.254</td>
<td>0.2</td>
<td>0.2</td>
</tr>
<tr>
<td>( n )</td>
<td>0.04</td>
<td>0.085</td>
<td>0</td>
</tr>
<tr>
<td>( l )</td>
<td>-0.4</td>
<td>-0.6</td>
<td>-0.3</td>
</tr>
<tr>
<td>( c_0 ) (J Kg(^{-1})K(^{-1}))</td>
<td>888</td>
<td>446</td>
<td>381</td>
</tr>
<tr>
<td>( q )</td>
<td>0.252</td>
<td>0.449</td>
<td>0.147</td>
</tr>
<tr>
<td>( K_0 ) (W m(^{-1})K(^{-1}))</td>
<td>163</td>
<td>55.4</td>
<td>387</td>
</tr>
<tr>
<td>( p )</td>
<td>0.437</td>
<td>-0.405</td>
<td>-0.081</td>
</tr>
<tr>
<td>( \rho ) (Kg m(^{-3}))</td>
<td>2700</td>
<td>7800</td>
<td>8950</td>
</tr>
<tr>
<td>( \beta )</td>
<td>0.95</td>
<td>0.95</td>
<td>0.95</td>
</tr>
</tbody>
</table>
in terms of a flow

where \( U \) is a velocity and

where the norma

are \( O(1) \) quantities obtained by differentiation with respect to \( \varepsilon \). \( v_2 \) is negligible compared to \( v_1 \) and \( \theta \) is the deformation tensor:

\[
d_{11} = (U\sigma_{11} + (\delta d_{11} + \delta d_{12} + \delta d_{13})/\varepsilon)
\]

To leading order

In addition, the equality

Next, we proceed to describe the semi-stagnation of \( w \). To this end we introduce the following equations of \( w \) describing by (17) and (18) the leading order in \( \varepsilon \):

are \( O(1) \). In turn

Inserting (29) into (17), (18), and (19) together with leading order in \( \varepsilon \), we obtain

where

Equation (21) is satisfied by writing the velocity field as

\[
v_{1,1} + v_{2,1} = (s_{11,1} + s_{12,2} + \delta_{1})/\rho, \\
v_{2,2} = (s_{21,1} + s_{22,2} + \delta_{2})/\rho, \\
\theta = (q_{1,1} + q_{2,2})/\rho C + \beta (s_{11}d_{11} + s_{22}d_{22} + 2s_{12}d_{12})/\rho C, \\
w_{1,1} + v_{2,1} = s_{11}d_{11} + s_{22}d_{22} + 2s_{12}d_{12}, \\
v_{1,1} + v_{2,2} = 0.
\]
Dynamic boundary layers and shear bands

\[ v_1 = \psi_{,2}, \quad v_2 = -\psi_{,1}, \tag{22} \]

in terms of a flow potential \( \psi \). Introduce the scaled variables

\[ x_1' = \frac{x_1}{L}, \quad x_2' = \frac{x_2}{\varepsilon L}, \quad t' = \frac{Ut}{L}, \quad \psi' = \frac{\psi}{\varepsilon UL}, \tag{23} \]

where \( U \) is a velocity characteristic of the free flow. Then, the velocity field (22) takes the form

\[ v_1 = Uv_1', \quad v_2 = \varepsilon Uv_2', \tag{24} \]

where the normalized velocities

\[ v_1' = \psi_{,2}', \quad v_2' = -\psi_{,1}', \tag{25} \]

are \( O(1) \) quantities. Here and subsequently, the notation \((\cdot)'\) is used to denote partial differentiation with respect to \( x' \). It follows from (24) that, within the boundary layer, \( v_2 \) is negligible compared to \( v_1 \). Similarly, we can scale the components of the rate of deformation tensor, with the result

\[ d_{11} = \frac{1}{2}(U/L)\psi_{,11}', \quad d_{22} = (U/L)\psi_{,22}', \quad d_{12} = \frac{1}{2}(U/L)\left(\frac{1}{\varepsilon} \psi_{,12}' + \varepsilon \psi_{,21}'\right). \tag{26} \]

To leading order in \( \varepsilon \), \( d_{11} \) and \( d_{22} \) are negligible and

\[ d_{12} \sim \frac{1}{2\varepsilon}(U/L)\psi_{,12}'. \tag{27} \]

In addition, the effective strain rate reduces to

\[ \dot{\gamma} \sim \frac{1}{\varepsilon}(U/L)|\psi_{,12}'|. \tag{28} \]

Next, we proceed to scale (20) governing the evolution of the hardening variable \( w \). To this end we start by assuming that \( \theta \) and \( w \) are quantities of order \( \varepsilon^\lambda \) and \( \varepsilon^\mu \), respectively, where \( \lambda \) and \( \mu \) are as yet unknown coefficients. Additionally, let \( T \) and \( W \) denote characteristic values of \( \theta \) and \( w \), respectively. Then, the scaled variables

\[ \theta' = \varepsilon^{-\lambda} \frac{\theta}{T}, \quad w' = \varepsilon^{-\mu} \frac{w}{W}, \tag{29} \]

are of \( O(1) \). In terms of scaled variables, the left hand side of (20) becomes

\[ w_{,t} + v_1 w_{,1} + v_2 w_{,2} = \varepsilon^\lambda W(U/L)(w_{,t}' + v_1 w_{,1}' + v_2 w_{,2}'). \tag{30} \]

Inserting (29) into (3) and taking into account the assumed power-law behavior described by (12), the components of the stress deviator tensor are found to be, to leading order in \( \varepsilon \),

\[ s_{11} \sim \varepsilon^{a+1} Ss_{11}', \quad s_{22} \sim \varepsilon^{a+1} Ss_{22}', \quad s_{12} \sim \varepsilon^{a} Ss_{12}', \tag{31} \]

where
\begin{align*}
S &= A(U/L)^m W^n T^l, \\
S_1 &= |v_1|^m |v_1|^n |w|^{n-1} \theta^l, \\
S_2 &= |v_2|^m |v_2|^n |w|^{n-1} \theta^l, \\
S_3 &= |v_3|^m |v_3|^n |w|^{n-1} \theta^l.
\end{align*}

and

\begin{equation}
\alpha = -m + \mu n + \lambda l.
\end{equation}

It is apparent from (31) that \( s_{12} \) is the dominant component of the stress deviator tensor. From (27) and (31) it follows that, to leading order, the plastic work rate reduces to

\begin{equation}
s_{ij}d_{ij} \sim 2s_{12}d_{12} \sim \varepsilon^{a-1}(U/L)Ss_{12}v_{1,2}.
\end{equation}

The exponent \( \mu \) is obtained by requiring the leading terms on both sides of (20) to be of the same order in \( \varepsilon \), which yields the relation

\begin{equation}
\mu = \alpha - 1.
\end{equation}

This completes the scaling of (20) which, to leading order, simplifies to

\begin{equation}
w_1' + v_1' w_1' + v_2' w_2' = Ss_{12}v_{1,2}.
\end{equation}

In a similar way, we proceed to scale the energy balance equation (19). The source term is proportional to the plastic work rate, the scaled form of which is given by (35). With the aid of (29a), the convective terms on the left hand side of (19) can be written in the form

\begin{equation}
\theta'(\theta_{,1} + \psi_{,1} + \psi_{,2}) = \varepsilon^{(q+1)}(U/L)T^{q+1}\theta'(\theta_{,1} + \psi_{,1} + \psi_{,2}).
\end{equation}

Finally, the order of the heat flux components can be ascertained by inserting (29a) into the generalized Fourier law (10), which yields

\begin{align*}
q_1 &\sim \varepsilon^{g+1} B T^{p+k} \frac{L^k}{q_1}, \\
q_2 &\sim \varepsilon^g B T^{p+k} \frac{L^k}{q_2},
\end{align*}

where

\begin{align*}
q_1' &= \theta' \frac{\partial \theta'}{\partial x_1} \frac{\partial r^{-1}}{\partial x_1}, \\
q_2' &= \theta' \frac{\partial \theta'}{\partial x_2} \frac{\partial r^{-1}}{\partial x_2}
\end{align*}

are \( O(1) \) quantities and

\begin{equation}
\beta = \lambda p + (\lambda - 1)k.
\end{equation}

From (39), the two terms in the divergence of the heat flux follow as

\begin{equation}
q_{1,1} \sim \varepsilon^{g+1} B T^{p+k} \frac{L^k}{q_1'},
\end{equation}

Clearly, \( q_{2,2} \) further dominates asymptotically as \( \varepsilon \rightarrow 0 \). Either the convective term on the right hand side of (43) by the boundary layer with the relative order \( \delta_0 \). Thus, conductive both cases are pertinent.

Assume that (35) and (43) we obt

which, in conjunction

whereupon (19)

If, on the contr

which, together

Then, (19) redu

Equations (45) material const:

(48) and comple

asymptotically
\[ q_{2,2} \sim \varepsilon^{\beta-1} B \frac{TT^{p-k}}{L^{k+1}} q_{2,2}^t. \]  \( (33) \)

Clearly, \( q_{2,2} \) furnishes the dominant contribution to the divergence of the heat flux.

At this point, the critical question arises whether heat conduction or convection dominates asymptotically. In general, it is not possible to balance all three exponents of \( \varepsilon \) when (33), (38) and (39) are inserted into the energy equation (19). Therefore, either the convection term (38) or the conduction term (33) must dominate asymptotically as \( \varepsilon \to 0 \). In the former case, a balance is established between the rate of heat generation by plastic dissipation and the rate at which cold material is convected into the boundary layer. In the latter case, the heat generated diffuses out of the boundary layer by conduction. Which heat transport mechanism dominates is determined by the relative order of the convection and conduction terms in the energy equation. Thus, convection (convection) is dominant if \( \beta - 1 < \lambda(q+1) \) (\( \beta - 1 > \lambda(q+1) \)). Since both cases are possible, we proceed to analyze them in turn.

Assume that conduction is dominant. Then, matching the exponents of \( \varepsilon \) in (35) and (33) we obtain

\[ \alpha = \beta, \]  \( (34) \)

which, in conjunction with (36), determines \( \lambda \) and \( \mu \) as functions of \( k, l, m, n \) and \( p \).

The result is

\[ \lambda = \frac{m+n+k(n-1)}{(p+k)(n-1)+l}, \quad \mu = \frac{(p+k)(m+1)-(k+1)l}{(p+k)(n-1)+l}, \]  \( (35) \)

whereupon (19) reduces to

\[ -q_{2,2}^t = \frac{\beta S(U/L)}{(B/L)T^\beta(T/L)} \varepsilon^{\beta/12} \varepsilon_{12}^t v_{1,2}^t. \]  \( (36) \)

If, on the contrary, we assume convection to be dominant, balancing the exponents of \( \varepsilon \) in (35) and (38) results in the relation

\[ \alpha = \lambda(1+q), \]  \( (37) \)

which, together with (36), gives

\[ \lambda = \frac{m+n}{(q+1)(n-1)+l}, \quad \mu = \frac{(m+1)(q+1)-l}{(q+1)(n-1)+l}. \]  \( (38) \)

Then, (19) reduces to

\[ \theta^\beta(\theta_{12}^t + \theta_{1,1'}^t + \theta_{2,2'}^t) = \frac{\beta S}{\rho CT^{\beta+1}} \varepsilon_{12}^t v_{1,2}^t. \]  \( (39) \)

Equations (35) and (38) render the scaling (29) of \( \theta \) and \( w \) determinate. For given material constants the exponents \( \beta - 1 \) and \( \lambda(q+1) \) can be computed from (35) and (38) and compared to determine whether convection or conduction is dominant asymptotically.
Finally, we proceed to scale the equations of linear momentum balance, (17) and (18). Using the normalization (24), the forces of inertia can be readily written in the form

\[ v_{1,v} + v_{1,v} + v_{2,v} = \frac{U^2}{L} (v_{1,v'} + v_{1,v'_{1,v''}} + v_{2,v'_{1,v''}}), \]

(50)

\[ v_{2,v} + v_{1,v} + v_{2,v} = \frac{U^2}{L} (v_{1,v'} + v_{1,v'_{1,v''}} + v_{2,v'_{1,v''}}). \]

(51)

To leading order in \( \varepsilon \), the derivatives of the components of the stress deviator tensor follow from (33) as

\[ s_{1,1,v} \sim \varepsilon^{s+1} \frac{S}{L} s_{1,1,v'}, \quad s_{1,2,v} \sim \varepsilon^{s-1} \frac{S}{L} s_{1,2,v'}, \quad s_{21,1,v} \sim \varepsilon^{s+2} \frac{S}{L} s_{21,1,v'}, \quad s_{22,2,v} \sim \varepsilon^{s+2} \frac{S}{L} s_{22,2,v'}. \]

(52)

Clearly, \( s_{1,1,v} \) is negligible compared to \( s_{1,2,v} \) and, consequently, (17) reduces to

\[ v_{1,v'} + v_{1,v'_{1,v''}} + v_{2,v'_{1,v''}} = \varepsilon^{s-1} \frac{S}{\rho U^2} s_{1,2,v'} + p'_{1,v'}, \]

(53)

where we have set

\[ p' = \frac{p}{\rho U^2} \]

(54)

which renders \( p_{1,v} \) of the same order in \( \varepsilon \) as the inertia forces.

At this point, we note that the value of \( \alpha = 1 \) is incompatible with (36) in general. Since the various normalized terms in (53) are of \( O(1) \), it therefore follows that \( \varepsilon^{s-1} S/\rho U^2 \) must itself be of \( O(1) \). Without loss of generality, we can set \( \varepsilon^{s-1} S/\rho U^2 = 1 \). This identification provides a physical definition for the small parameter \( \varepsilon \), namely,

\[ \varepsilon = R^{1/(1-s)}, \]

(55)

where

\[ R = \frac{\rho U^2}{S}. \]

(56)

We can now write (17) in the scaled form

\[ v_{1,v'} + v_{1,v'_{1,v''}} + v_{2,v'_{1,v''}} = s_{1,2,v'} + p'_{1,v'}, \]

(57)

whereas from (51), (52) and (54) the scaled form of (18) is found to be

\[ p_{1,2,v'} = 0. \]

(58)

The scaling of (17)–(21) is thus completed. It bears emphasis that, as a by-product of the scaling argument, a precise physical definition of the small parameter \( \varepsilon \) is obtained.

For later reference, we proceed to collect the boundary layer equations derived in the conduction and in the convection in the normalized boundary layer equations derived in the foregoing.

Upon normalization, we can write

\[ \tilde{x}_1, \quad \tilde{x}_2, \quad \tilde{x}_3, \quad \tilde{x}_4, \quad \tilde{x}_5. \]
Dynamic boundary layers and shear bands

In the foregoing. In applications, it proves convenient to work with the following dimensionless variables:

\[
\begin{align*}
\dot{x}_1 &= \frac{x_1}{L}, \quad \dot{x}_2 = \frac{x_2}{L}, \quad \bar{t} = \frac{t}{L}, \quad \bar{\psi} = \frac{\psi}{UL}, \quad \bar{v}_1 = \frac{v_1}{U}, \\
\bar{v}_2 &= \frac{v_2}{U}, \quad \bar{w} = \frac{w}{W}, \quad \bar{\theta} = \frac{\theta}{T}, \quad \bar{\rho} = \frac{\rho}{\rho U^2}.
\end{align*}
\]

Upon normalization, the mechanical boundary layer equations become

\[
\frac{\partial \bar{v}_1}{\partial \bar{t}} + \bar{v}_1 \frac{\partial \bar{v}_1}{\partial \bar{x}_1} + \bar{v}_2 \frac{\partial \bar{v}_1}{\partial \bar{x}_2} = \frac{1}{\mathcal{R}} \frac{\partial \bar{s}_{12}}{\partial \bar{x}_2} + \frac{\partial \bar{\rho}}{\partial \bar{x}_1},
\]

\[
\frac{\partial \bar{\rho}}{\partial \bar{x}_2} = 0,
\]

\[
\frac{\partial \bar{v}_1}{\partial \bar{x}_1} + \frac{\partial \bar{v}_2}{\partial \bar{x}_2} = 0,
\]

\[
\bar{s}_{12} = \left| \frac{\partial \bar{v}_1}{\partial \bar{x}_2} \right| -\bar{\rho} \bar{\theta},
\]

\[
\frac{\partial \bar{w}}{\partial \bar{t}} + \bar{v}_1 \frac{\partial \bar{w}}{\partial \bar{x}_1} + \bar{v}_2 \frac{\partial \bar{w}}{\partial \bar{x}_2} = \mathcal{P} \bar{s}_{12} \frac{\partial \bar{v}_1}{\partial \bar{x}_2}.
\]

Similarly, the reduced energy equation is

\[
-\frac{\partial q_2}{\partial \bar{x}_2} = \mathcal{J}_d \bar{s}_{12} \frac{\partial \bar{v}_1}{\partial \bar{x}_2},
\]

\[
q_2 = \bar{\rho} \left| \frac{\partial \bar{\theta}}{\partial \bar{x}_2} \right|^{\kappa-1} \frac{\partial \bar{\theta}}{\partial \bar{x}_2}
\]

in the conduction-dominated case, and

\[
\bar{\theta} \left( \frac{\partial \bar{\theta}}{\partial \bar{t}} + \bar{v}_1 \frac{\partial \bar{\theta}}{\partial \bar{x}_1} + \bar{v}_2 \frac{\partial \bar{\theta}}{\partial \bar{x}_2} \right) = \mathcal{J}_c \bar{s}_{12} \frac{\partial \bar{v}_1}{\partial \bar{x}_2}
\]

in the convection-dominated case.

The following dimensionless numbers arise when formulating the preceding normalized boundary layer equations

\[
\mathcal{R} = \frac{\rho U^2}{S},
\]

\[
\mathcal{P} = \frac{S}{W},
\]
\[ T_a = \frac{\beta S(U/L)}{(B/L)T^a(T/L)^k} \]
\[ T_c = \frac{\beta S(U/L)}{\rho C T^{a+1}(U/L)} \]

The numerators of \( T_a \) and \( T_c \) define characteristic values of the rate of heat generation due to plastic dissipation, while the denominators are characteristic values of the rate of heat transport by conduction and convection, respectively. Therefore, the numbers \( T_a \) and \( T_c \) quantify the ability of conduction and convection, respectively, to extract from the boundary layer the heat generated by plastic dissipation. The ratio

\[ \mathcal{P} = \frac{T_c}{T_a} = \frac{(B/L)T^a(T/L)^k}{\rho C T^{a+1}(U/L)} \]

measures the relative importance of heat conduction and convection as heat transport mechanisms and can, therefore, be regarded as a generalized Stanton number. Neither \( T_a \) nor \( T_c \) appear to have been named in the literature. Finally, \( \mathcal{P} \) is a ratio of characteristic values of the rate of plastic dissipation and the rate at which plastic work is convected by the flow.

The dimensionless number \( R \) in (69) plays the role of a generalized Reynolds number. Indeed, in the Newtonian case, which is recovered by setting \( m = 1, n = l = 0 \), \( R \) reduces to \( UL/\nu \), where \( \nu = \eta/\rho \) is the kinematic viscosity and \( \eta = \sigma_0/\gamma_0 \) is the viscosity. This is the conventional definition of the Reynolds number for Newtonian fluids. As in the Newtonian case, the generalized Reynolds number represents the ratio between characteristic values of inertia and viscous stresses. In view of relation (55), the assumption \( \epsilon \ll 1 \) leading to the boundary layer equations is now seen to correspond to the requirement that \( R \gg 1 \). In other words, the asymptotic regime in which the boundary layer equations apply is that of very high Reynolds numbers.

The asymptotic structure of the flow in this regime is well-known from singular perturbation theory. Away from the boundary layer the term \( s_{12,2} \) in (61) can be neglected. This determines the solution outside the boundary layer, or outer solution. The outer solution does not satisfy all displacement boundary conditions in general. Physically, compatibility is achieved by the development of a narrow boundary layer where the solid experiences high rates of shearing. The solution in the boundary layer, or inner solution, obeys the boundary layer equations (61)–(68). In general, the inner solution contains arbitrary constants which are determined by asymptotic matching to the outer solution.

### 2.3. Conduction and convection sublayers

The preceding analysis shows that either conduction or convection must dominate asymptotically to the exclusion of the other. Which mechanism dominates can be ascertained by comparing the exponents \( 9 - 1 \) and \( \lambda(q + 1) \). The numerical values of these exponents are used in the calculations (34), (36), (41) and the resulting nondimensionalization which shows that the exponents are considerably different. Conduction is dominant and \( \epsilon < \lambda(q + 1) \), in accordance with the deformed typical solution.

However, even for \( \mathcal{P} \), (73), is exceed the normalized energy in the sublayer, characterizing it as a sublayer containing nested transition with the innermost layer matching. In addition, this follows from the asymptotic calculations.

The requisite terms in (74) are given by: With the thinness of the sublayer, the solution at \( x_2 = \) where, here and solution, and the asymptotic matching to (74) is readily
Table 2. Orders of the conduction and convection terms in the energy equation

<table>
<thead>
<tr>
<th>Assuming</th>
<th>Exponent</th>
<th>Pure Al</th>
<th>Iron</th>
<th>OFHC Cu</th>
</tr>
</thead>
<tbody>
<tr>
<td>convection is dominant</td>
<td>( g - 1 )</td>
<td>-1.23</td>
<td>-1.21</td>
<td>-1.16</td>
</tr>
<tr>
<td></td>
<td>( \lambda(q + 1) )</td>
<td>-0.226</td>
<td>-0.208</td>
<td>-0.159</td>
</tr>
<tr>
<td>conduction is dominant</td>
<td>( g - 1 )</td>
<td>-1.46</td>
<td>-1.67</td>
<td>-1.40</td>
</tr>
<tr>
<td></td>
<td>( \lambda(q + 1) )</td>
<td>0.469</td>
<td>0.812</td>
<td>0.753</td>
</tr>
</tbody>
</table>

These exponents for three different metals are shown in Table 2. The material constants used in the calculations are listed in Table 1. If the exponents are computed from (34), (36), (41) and (47) under the assumption that convection is dominant, the resulting numerical values are found to violate the condition \( g - 1 > \lambda(q + 1) \), a contradiction which shows that convection cannot be dominant. If, on the other hand, the exponents are computed from (34), (36), (41) and (44) under the assumption that conduction is dominant, the resulting numerical values satisfy the condition \( g - 1 < \lambda(q + 1) \), in accordance with the assumption. These results suggest that, for highly deformed typical metals, conduction dominates asymptotically as \( \varepsilon \to 0 \).

However, even when conduction dominates asymptotically, a more detailed analysis reveals that both convection and conduction may operate concurrently within the boundary layer. A common occurrence in many solid flows is that the Stanton number \( \mathcal{S} \), (73), is exceedingly small. Under these conditions, the heat conduction term in the normalized energy equation is multiplied by a small number and, while conduction dominates at the center of the boundary layer, it does so over an exceedingly narrow region, or conduction sublayer. In the remainder of the boundary layer, or convection sublayer, conduction is negligible and convection dominates. Thus, flows characterized by a small Stanton number, \( \mathcal{S} \ll 1 \), give rise to “complex” boundary layers containing nested sublayers. The convection sublayer plays the role of outer solution to the innermost conduction sublayer. Both solutions are related by asymptotic matching. In addition, the convection sublayer is to be matched to the outer flow. It therefore follows that both the conduction and convection solutions are required for a full characterization of the boundary layer structure.

The requisite matching of the conduction and convection sublayers can be effected as follows. Within the conduction sublayer (66) must be satisfied. Owing to the thinness of the sublayer, all terms in (66) other than the rapidly varying conduction term can be treated as constants and their values computed from the convection solution at \( \hat{x}_2 = 0 \). The energy equation (66) thus reduces to

\[
- \frac{\partial q_2^1}{\partial \hat{x}_2} = \mathcal{S}_0 \left[ \hat{s}_{12} \frac{\partial \hat{w}^v}{\partial \hat{x}_2} \right]_{\hat{x}_2 = 0} \equiv \mathcal{S}_d \hat{w}_0^v(\hat{x}_1),
\]

where, here and subsequently, the label \( v \) is used to denote the outer convection solution, and the label \( d \) to denote the inner conduction solution. The general solution to (74) is readily found to be
\[ \tilde{\theta}^k = \left[ \frac{k + p}{k + 1} (\mathcal{F}_{\delta} \tilde{\theta}_{\delta}^k)^{1/k} \right]^{k/(k + p)} \left( \tilde{b} + \bar{a} \bar{x}_2 - \bar{x}_2^{k+1}/k \right)^{1/(k + p)}, \] (75)

where \( \bar{a} \) and \( \tilde{b} \) are functions of \( \bar{x}_1 \) to be determined. In the special case of linear heat conduction, \( k = 1 \), and constant heat capacity, \( p = 0 \), (75) specializes to the expected quadratic variation of \( \tilde{\theta} \) with \( \bar{x}_2 \).

To determine the unknown functions \( \tilde{b}(\bar{x}_1) \) and \( \tilde{a}(\bar{x}_1) \), let \( \bar{x}_2 = \tilde{a}(\bar{x}_1) \) be the boundary between the conduction and convection sublayers. Then, two constraints are furnished by the requirement that the temperature and the heat flux be continuous at \( \bar{x}_2 = \tilde{a}(\bar{x}_1) \), namely,

\[ \tilde{\theta}^k(\bar{x}_1, \tilde{a}(\bar{x}_1)) = \tilde{\theta}^k(\bar{x}_1, \tilde{a}(\bar{x}_1)), \] (76)

\[ \tilde{q}_2^k(\bar{x}_1, \tilde{a}(\bar{x}_1)) = \tilde{q}_2^k(\bar{x}_1, \tilde{a}(\bar{x}_1)), \] (77)

where \( \tilde{\theta}^k(\bar{x}_1, \bar{x}_2) \) and \( \tilde{q}_2^k(\bar{x}_1, \bar{x}_2) \) are assumed known. An additional constraint follows from the boundary conditions. As an example, consider an external boundary layer (that is, a boundary layer which develops along a contact) with a mixed boundary condition of the form

\[ q_2 + \kappa \theta = h \quad \text{at} \quad \bar{x}_2 = 0, \] (78)

where \( \kappa \) is a heat-transfer coefficient and the function \( h(x_1) \) is prescribed. Dirichlet and Neumann boundary conditions follow as the limits of \( h \to \infty \) and \( h \to 0 \), respectively. Upon normalization, (78) becomes

\[ \tilde{q}_2^k + \mathcal{K} \tilde{\theta}^k = \tilde{h} \quad \text{at} \quad \bar{x}_2 = 0, \] (79)

where the dimensionless number

\[ \mathcal{K} = \frac{\kappa T}{BT^p(T/L)^k} \] (80)

measures the relative conductances of the contact and the solid.

Equations (76), (77) and (79) completely determine \( \tilde{b}(\bar{x}_1) \), \( \tilde{a}(\bar{x}_1) \) and \( \tilde{a}(\bar{x}_1) \). It bears emphasis that, for fixed \( \bar{x}_1 \), these equations are algebraic. In cases where the asymptotic structure of the convection fields as \( \bar{x}_2 \to 0 \) is known, \( \tilde{b}(\bar{x}_1) \) and \( \tilde{a}(\bar{x}_1) \) can be determined explicitly. In the simple but important case of linear heat conduction, \( k = 1 \), and constant heat capacity, \( p = 0 \), (76), (77) and (79) define a linear system of equations in the unknowns \( \tilde{a}^i(\bar{x}_1) \), \( \tilde{b}^i(\bar{x}_1) \) and \( \tilde{a}^i(\bar{x}_1) \).

Other boundary conditions can be treated similarly. A case in point is that of internal boundary layers. In this case, two solutions of the form (75) apply in the regions \( \bar{x}_2 > 0 \) and \( \bar{x}_2 < 0 \), respectively. The six unknowns \( a^\pm(\bar{x}_1) \), \( b^\pm(\bar{x}_1) \) and \( \tilde{a}(\bar{x}_1) \) follow by matching to the upper and lower convection fields—which gives four equations of the type (76) and (77)—and by imposing the continuity conditions

\[ \tilde{\theta}^k(\bar{x}_1, 0^+) = \tilde{\theta}^k(\bar{x}_1, 0^-), \] (81)
\[ \varphi_2^+(\bar{x}_1, \theta^+) = \varphi_2^-(\bar{x}_1, \theta^-). \]  

(82)

Examples of this procedure are presented in the next section.

3. STEADY SIMILARITY SOLUTIONS

Similarity methods (Rosenhead, 1963) constitute a powerful tool for obtaining semi-analytical solutions of the boundary layer equations developed in the foregoing. An appealing feature of similarity solutions is that their determination requires the solution of a system of ordinary differential equations. This system can conveniently be solved by numerical integration, and the complete two-dimensional fields recovered. In the present section, attention is restricted to steady boundary layers. Transient solutions will be considered in Section 5.

For present purposes it suffices to consider free flows in which the hydrostatic stress is uniform, i.e. \( \bar{\varphi}_i = 0 \). Free flows of the Falkner–Skan type (Falkner and Skan, 1930, 1931), in which the hydrostatic stress gradient varies as a power of \( \bar{x}_i \), can be treated similarly. For definiteness, we envision a semi-infinite plate occupying the half-plane \( \bar{x}_1 \geq 0 \) to which a velocity \( U \) is imparted on the lower half of the boundary, i.e. on \( \bar{x}_1 = 0, \bar{x}_2 < 0 \), Fig. 1. The free flow appropriate to this problem is

\[ v_1 = 0, \quad v_2 = 0, \quad \theta = T, \quad w = W, \quad \text{for} \ x_2 > 0, \]  

(83)

and

\[ v_1 = U, \quad v_2 = 0, \quad \theta = T, \quad w = W, \quad \text{for} \ x_2 < 0. \]  

(84)

In addition, the boundary layer solution is subject to velocity and traction continuity conditions on the \( \bar{x}_1 \) axis. It follows from (83) and (84) that the origin is a singular point of the flow and, consequently, the boundary layer theory may be expected to break down in its vicinity.

Solutions to the boundary layer equations can be obtained by introducing the similarity variable

\[ \zeta = \bar{x}_2 \bar{x}_1^{-a}. \]  

(85)

The normalized stream function, temperature and plastic work fields are then expressed as

\[ \bar{\psi} = \bar{x}_1^a f(\zeta), \quad \bar{\theta} = \bar{x}_1^b g(\zeta), \quad \bar{w} = \bar{x}_1^c h(\zeta), \]  

(86)

where the functions \( f, g \) and \( h \) and the exponents \( a, b \) and \( c \) are to be determined. The velocity and shear stress fields follow from (86) in the form

\[ \bar{v}_1 = f', \quad \bar{v}_2 = -a \bar{x}_1^{-1}(f'-f'' \zeta), \quad \bar{\sigma}_{12} = \bar{x}_1^c \operatorname{sgn}(f''), \]  

(87)

where

\[ \tau = |f'|^m g'/h^a, \]  

(88)
and

\[ d = -am + cn + bl. \] (89)

Inserting representation (86) into the boundary layer equations and balancing out powers of \( \tilde{x}_1 \), the following system of ordinary differential equations is obtained

\[ -aff'' = \frac{1}{\beta} \tau \left( m \frac{f''}{|f''|} + n \frac{h'}{h} + l \frac{g'}{g} \right), \] (90)

\[ chf' - afh' = \delta \tau |f''|, \] (91)

where \((\cdot)'\) denotes differentiation with respect to \( \zeta \). In the conduction-dominated case, the energy equation reduces to

\[ -g^a g^b \left( p \frac{g'}{g} + k \frac{g''}{g'} \right) = \delta \tau |f''|, \] (92)

and the characteristic exponents are found to be

\[ a = \frac{p + k - l}{(p + k)(m + 1) - l(k + 1)}, \quad b = -\frac{m - k}{(p + k)(m + 1) - l(k + 1)}, \quad c = 0. \] (93)

It follows from these relations that the plastic work field is similar (that is to say, it only depends on \( \tilde{x}_1 \) and \( \tilde{x}_2 \) through the similarity variable), unlike the temperature and shear stress fields which vary as \( \tilde{x}_1^x \) and \( \tilde{x}_2^y \), respectively, at \( \tilde{x}_2 = 0 \). In the convection-dominated case the energy equation likewise reduces to

\[ bg^{1+a}|f'| - afg^b g' = \delta \tau |f''|, \] (94)

while the characteristic exponents take the values

\[ a = \frac{1}{m + 1}, \quad b = 0, \quad c = 0. \] (95)

Consequently, both the temperature and plastic work fields are similar in this case. By contrast, the shear stress field is not similar in general. The boundary conditions at infinity are obtained by matching to the outer field (83) and (84). This gives

\[ f' \to 0, \quad g \to 1, \quad h \to 1, \quad \text{as} \ \zeta \to \infty, \] (96)

and

\[ f' \to 1, \quad g \to 1, \quad h \to 1, \quad \text{as} \ \zeta \to -\infty. \] (97)

Assume that the boundary layer structure is of the complex type discussed in Section 2.3, with convection dominant everywhere except for a narrow conduction-dominated sublayer at the core. The convection sublayer follows from (88), (90), (91) and (94). For definiteness, we shall seek solutions such that \( \tilde{v}_2 = 0 \) on the \( \tilde{x}_1 \) axis. Since \( \tilde{x}_2 = 0 \) necessitates \( \zeta = 0 \), it follows from (87b) that

\[ f(0) = 0. \] (98)
In view of (90), this condition in turn implies that \( \tau'(0) = 0 \) and, consequently, \( \tau \) attains a maximum at \( \zeta = 0 \). The \( \bar{x}_1 \) axis may therefore be regarded as the centerline of the boundary layer. An additional consequence of (98) is that \( g' \) and \( h' \) must necessarily be singular at \( \zeta = 0 \), as is evidenced by taking the limit of \( \zeta \to 0 \) in (91) and (94). Therefore, neither \( g \) nor \( h \) can be continued analytically across the \( \bar{x}_1 \) axis. This requires the solutions in the upper and lower half planes to be computed separately. The requisite velocity and traction compatibility conditions at \( \zeta = 0 \) are

\[
f'(0^+) = f'(0^-), \quad \tau(0^+) = \tau(0^-).
\]

The upper and lower solutions can now be computed numerically, e.g., by a shooting procedure. Once the convection-dominated fields are known, the conduction sublayer can be fitted by the method outlined in Section 2.3.

Certain salient geometrical features of the solution, such as the variation of the boundary layer “thickness” with \( \bar{x}_1 \), can be determined analytically. Because the boundary layer solution approaches the outer solution asymptotically, the transition from one to the other is gradual, and the definition of a boundary layer thickness is, to some degree, a matter of convention. A simple definition is to identify the thickness \( \delta_u(\bar{x}_1) \) of the mechanical boundary layer with the ordinate \( \bar{x}_2 \) at which \( \bar{v}_1 \) attains a prescribed fraction \( \chi \) of the free-flow velocity, i.e.,

\[
\bar{v}_1(\bar{x}_1, \delta_u(\bar{x}_1)) = \chi
\]

with \( \chi \) smaller than, but close to, 1. Using representation (87a), the definition of the boundary layer thickness becomes

\[
f'(\zeta_u(\bar{x}_1)) = \chi,
\]

where we have set

\[
\zeta_u(\bar{x}_1) \equiv \delta_u(\bar{x}_1) \bar{x}_1^{-\alpha}.
\]

Differentiating both sides of (101) with respect to \( \bar{x}_1 \) and taking (102) into account yields the relation

\[
f''(\zeta_u(\bar{x}_1)) \bar{x}_1^{-\alpha} \left( \frac{d\delta_u}{d\bar{x}_1} - \frac{a}{\bar{x}_1} \delta_u \right) = 0.
\]

This equation is satisfied by setting

\[
\frac{d\delta_u}{d\bar{x}_1} = a \frac{\delta_u}{\bar{x}_1},
\]

This simple differential equation determines the mechanical boundary layer thickness \( \delta_u \). The solution is

\[
\delta_u(\bar{x}_1) = Z \bar{x}_1,
\]
where $Z$ follows from the condition $f'(Z) = \chi$. As may be noted, when the boundary layer is convection dominated, $a = 1/(m+1)$ and $\delta_a$ is a monotonically increasing function of $\bar{x}_1$.

3.1. Example

Consider, by way of example, the case of a copper plate which develops a steady internal boundary layer such as described in the foregoing. As shall become apparent in Section 4, the material constants listed in Table 1, which correspond to relatively large strains, lead to unstable behavior. This precludes the formation of a steady boundary layer. Instead, in the present example we shall take $n = 0.1$ and $l = -0.1$, which result in stable behavior. A comparatively high rate of hardening is indeed characteristic of the early stages of deformation of copper in which the material may be expected to behave stably.

In calculations, we employ a fourth-order Runge–Kutta method to integrate (90) from boundary conditions at $\zeta = 0$. Integration into the positive and negative $\zeta$-directions is performed separately based on assumed boundary values of $f''$ and $f''$. Simultaneously, (91) and (94) are integrated by the forward–Euler method to determine $g$ and $h$. Since the remote values of these functions are known, the integration of (91) and (94) can conveniently be effected from the free flow, i.e. from $\zeta = \pm \infty$ towards the origin. The unknown values $f''(0^\pm)$ are determined iteratively so as to match the remote boundary conditions $f'(\infty) = 1$ and $f'(\infty) = 0$. Finally, continuity of tractions at the origin is achieved by iteration on $f'(0)$.

The impact velocity $U$ is set at 544 m s$^{-1}$. Lengths are measured in units of the characteristic dimension $L = U/\gamma_0 = 1.1$ mm, which corresponds to the gage length over which the reference strain rate is attained for the prescribed impact velocity. In addition, we set the free flow plastic work and temperature to the reference values adopted in Table 1, i.e. $W = \omega_0 = \sigma_0\gamma_0$ and $T = \theta_0$, which results in a Reynolds number $\mathcal{R} = \rho U^2/\sigma_0 = 10$, and dimensionless numbers $\mathcal{P} = \sigma_0^2/W = 1/\gamma_0 = 5$, $\mathcal{F}_0 = \beta\sigma_0/\rho c_0\gamma_0 = 0.25$, and $\mathcal{F}_d = \beta\sigma_0\gamma_0 L^2/K_0 T = 1305$. We verify that the Stanton number $\mathcal{S} = \mathcal{F}_0/\mathcal{F}_d = 0.000192$ is indeed small, which implies that the boundary layer has the complex structure described in Section 2.3.

The profiles of the similarity solution computed by numerical integration at intervals of $\Delta \zeta = 0.005$ are shown in Fig. 2. As expected, the velocity profile effects a smooth transition from its limiting value of one at large and negative $\zeta$ to its limiting value of 0 at large and positive $\zeta$, Fig. 2(a). The shear stress profile attains a maximum $\tau = 1.34$ at $\zeta = 0$, and decays monotonically but asymmetrically to zero away from the layer, Fig. 2(b). The convection temperature profile, shown as the dotted line in Fig. 2(c), diverges to infinity at $\zeta = 0$, as expected. However, the conduction sublayer renders the temperatures bounded everywhere. The plastic work distribution also diverges to infinity at $\zeta = 0$, Fig. 2(d). This may be regarded as an artifact of the steady solution. The transient solutions presented in Section 5 are devoid of this unphysical behavior.

The shear stress and conduction temperature fields predicted by the theory are not similar and, consequently, they vary with $\bar{x}_1$ along the centerline $\bar{x}_2 = 0$. This variation is displayed in Fig. 3(a). The shear stress decreases downstream monotonically as

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\( x_{j}^{0.33}. \) The temperature exhibits a moderate rise from a value of 733°C near the origin
to a value of 910°C at 5.5 cm from the root. Also shown in Fig. 4 are the \( x_{j} \)-variations
of the widths of the convection and conduction sublayers. As expected, both spread
downstream from the origin, and do so asymmetrically. The thinness of the conduction
sublayer relative to the convection sublayer is particularly noteworthy.

Fig. 2. Steady boundary layer in a copper plate impacted by a flat-ended rigid projectile. Profiles of (a)
velocity; (b) shear stress; (c) temperature; (d) plastic work.
Fig. 3. Steady boundary layer in a copper plate impacted by a flat-ended rigid projectile. Variation of temperature and shear stress at $\tilde{x}_2 = 0$ with distance from the root.

Fig. 4. Steady boundary layer in a copper plate impacted by a flat-ended rigid projectile. Variation with distance from the root of: (a) width of convection sublayer and (b) width of conduction sublayer.

4. ASYMPTOTIC ANALYSIS AND STABILITY

While similar boundary layer solutions cannot be obtained analytically, even in the newtonian case, their asymptotic structure as $\zeta \to 0$ can be characterized readily. The asymptotic result the asymptotic analysis of stability. We shall focus on the existence of layers. An analytic solutions in unstable systems. We begin by making the existence of a layer to a domain. Convection and proceed to analyze constitutive production and norm methods of analysis by Molinari and others.

4.1. Convection-

We begin by considering the boundary layer equation. We make the following approximation for the width of the boundary layer as $\zeta \to 0$:

$$\delta_c \sim \zeta$$

where $\tau_0, \tau_1$ and $\delta_c, \delta_1$ the above exponents and the norms at $\zeta = 0$. Finally, for conditions at infinity:

Inserting (10) of $\zeta$ gives

As expected, $\delta_0$ balancing power balances:

For most solids conditions, the
asymptotic results shed considerable light on the form of the solution. Interestingly, the asymptotic analysis also leads to the statement of simple conditions for material stability. We shall say that a material is stable when it can support steady boundary layers. An analysis due to Molinari and Clifton (1987) shows that one-dimensional solutions in unstable materials blow up in finite time. In light of this result, we take the existence of steady boundary layer solutions to imply material stability. In the analysis of Molinari and Clifton, solutions blow up by localizing to a band of zero thickness. Similarly, here instability manifests itself as the collapse of the boundary layer to a domain of zero thickness. By analogy to fluid mechanics, the collapsed layers may be regarded as vortex sheets.

Conditions for the existence of steady boundary layers can be inferred by examining the asymptotic form of the solutions near the boundary. For present purposes it suffices to consider the simplest of geometries, namely, the boundary layer at a rigid contact, as stability is solely dictated by constitutive behavior. As usual, the convection- and conduction-dominated cases require separate treatment, and we proceed to analyze them in turn. The analysis reveals the effect on stability of such constitutive properties as rate sensitivity, hardening, thermal coupling, heat conduction and non-local hardening. It is noteworthy that, despite the widely disparate methods of analysis, our results are in agreement with the stability conditions derived by Molinari and Clifton (1987) for the adiabatic case.

4.1. Convection-dominated case

We begin by considering the case in which heat conduction is negligible throughout the boundary layer and convection is the only operative heat-transfer mechanism. We make the following assumptions regarding the asymptotic structure of the boundary layer as $\zeta \to 0$

\[ f \sim \zeta^{\zeta}, \quad g \sim \zeta^{\zeta}, \quad h \sim \zeta^{\zeta}, \quad \tau \sim \tau_0 + \tau_1 \zeta^{\zeta}, \quad (106) \]

where $\tau_0$, $\tau_1$ and the exponents $\zeta_0$, $\zeta_1$ and $\zeta_3$ are to be determined. By way of motivation for the above expressions, it may be noted that (106a) is consistent with the boundary conditions $f(0) = f'(0) = 0$, provided that $\zeta > 1$. In addition, (106b) and (106c) reflects the expected singular behavior of $g$ and $h$ near the contact when $\zeta_0 < 0$ and $\zeta_3 < 0$. Finally, (106d) assigns a finite value of stress on the contact. The boundary conditions at infinity are irrelevant as regards the present analysis.

Inserting (106) into the hardening evolution equation (91) and balancing powers of $\zeta$ gives

\[ \varepsilon_h = -1. \quad (107) \]

As expected, $\varepsilon_h < 0$ and $h$ blows up as $\zeta^{-1}$ as the boundary is approached. Similarly, balancing powers of $\zeta$ in the energy equation yields

\[ \varepsilon_g = -\frac{1}{q+1}. \quad (108) \]

For most solids, $q > 0$, i.e. the specific heat increases with temperature. Under these conditions, the temperature tends to infinity as $\zeta^{-1/(q+1)}$ as $\zeta \to 0$, its distribution
becoming increasingly singular with decreasing $q$. It is interesting to note that the convection-dominated solution does not allow for the satisfaction of general thermal boundary conditions. If, for instance, the solid were in contact with a perfectly insulating substrate, a conduction sublayer would be required to accommodate the additional boundary condition $g'(0) = 0$.

Balancing powers of $\zeta$ in the stress–strain relation (88) and taking (107) and (108) into account gives

$$\varepsilon_f = 2 + \frac{\bar{n} + \bar{l}}{m},$$

(109)

where we have denoted

$$\bar{n} = \frac{n}{n+1}, \quad \bar{l} = \frac{l}{q+1}.$$  

(110)

Finally, $\varepsilon_c$ follows from the equation of linear momentum balance, (90), and (109), with the result

$$\varepsilon_c = 3 + 2 \frac{\bar{n} + \bar{l}}{m}.$$  

(111)

Equations (106)–(111) fully characterize the asymptotic behavior of the boundary layer as $\zeta \to 0$. In the special case of a Newtonian fluid, which is obtained by setting $m = 1$, $n = l = 0$, the familiar result $\varepsilon_f = 2$ is recovered from (109).

The condition for stability can be ascertained from the preceding asymptotic analysis. We expect the transition towards instability to be characterized by a progressive collapse of the steady solutions into layers of zero thickness or vortex sheets. Therefore, the boundary between stable and unstable behavior is characterized by velocity fields which asymptotically behave as $\bar{v}_1 \sim \text{constant}$. Since $\bar{v}_1 \propto f'$, this requires $\varepsilon = 1$, which, in view of (109), in turn necessitates $m + \bar{n} + \bar{l} = 0$. It is further noted that a well-defined asymptotic velocity profile is obtained if $\varepsilon > 1$. Hence, stability requires that

$$m + \bar{n} + \bar{l} > 0.$$  

(112)

Remarkably, this result coincides with the stability condition derived by Molinari and Clifton (1987) in a one-dimensional setting. Molinari and Clifton’s analysis also neglected inertia and heat conduction. The agreement between the two analyses underscores the fact that stability, as understood here, is not influenced by the geometry of the solid, or the distribution and history of the loads, but is strictly a consequence of constitutive behavior.

The presumption that the steady velocity profile collapses to a step function as the stability boundary (112) is approached is borne out by the numerical tests shown in Fig. 5. The choice of constants is as in the example of Section 3.1, except for $l$ which is set to $-0.5$. Under these conditions, $m + \bar{n} + \bar{l} = -0.144$ and the material is unstable. The numerical solution is artificially stabilized by virtue of the limited spatial resolution afforded by the mesh, but steadily collapses as the mesh size $\Delta \zeta$ is reduced,

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Fig. 5. Numerical velocity and shear stress profiles computed with decreasing increments of the similarity variable $\zeta$. (a) Unstable case, $m + \bar{\eta} + \bar{l} = -0.144$; and (b) stable case $m + \bar{\eta} + \bar{l} = 0.204$.

Fig. 5(a). The stable case exhibits no such mesh dependency and converges properly as the mesh is refined, Fig. 5(b).

4.2. Conduction-dominated case

Next we consider the case in which heat conduction is dominant. Under these conditions, the temperature field is bounded and (106b) no longer gives its correct asymptotic behavior. Instead, we assume that

$$g \sim g_0 + g_1 \zeta^\gamma.$$  \hspace{1cm} (113)

Evidently, values of $e_\gamma > 1$ are required for the boundary condition $g'(0) = 0$ to be
satisfied. The computation of constants and characteristic exponents proceeds as in the convection-dominated case. The result is

\[ \varepsilon_\theta = -1, \quad \varepsilon_\gamma = \frac{m(k+1)+\bar{n}}{km}, \quad \varepsilon_r = \frac{m+2\bar{n}}{m}, \quad \varepsilon_f = \frac{2m+\bar{n}}{m}. \] (114)

The range of stability is characterized by the condition \( \varepsilon_f > 1 \), which in view of (114d) necessitates

\[ m + \bar{n} > 0. \] (115)

Remarkably, (115) implies that, under conduction-dominated conditions, thermal softening has no effect on stability. By comparison to (112), it may be concluded that heat conduction compensates for—and eliminates—the destabilizing effect of thermal softening.

The above stability criteria further restrict the conditions under which complex boundary layers, i.e., those containing nested convection and conduction sublayers, may arise. It was noted in Section 2.3 that a necessary condition for complex boundary layers to be established is that the Stanton number \( \mathcal{S} \) be small. However, this condition by itself is not sufficient. Indeed, consider a material for which \( m + \bar{n} > 0 \) but \( m + \bar{n} + \tilde{\mu} < 0 \). These constitutive inequalities are indeed satisfied by the three metals listed in Table 1. For these materials, conduction-dominated layers are stable. By contrast, convection sublayers are unstable and collapse to a line. Consequently, convection-dominated layers cannot encompass conduction sublayers and, inevitably, the totality of the boundary layer is dominated by conduction.

4.3. Effect of non-local hardening

Non-local hardening models have been proposed as a vehicle for introducing a characteristic length into the constitutive description (Coleman and Hodgdon, 1985; Pijaudier-Cabot and Bazant, 1987; Vardoulakis and Aifantis, 1991, 1994; de Borst and Sluys, 1991; Sluys et al., 1993). Most of these models are phenomenological in nature, but some arise from micromechanical considerations (Vardoulakis and Aifantis, 1991, 1994). The presence of a characteristic length in the constitutive description is often found to prevent the collapse of adiabatic shear bands, which retain a well-defined structure.

In this section, we investigate the effect of non-local hardening on material stability. For purposes of illustration, we adopt a simple non-local hardening law of the type

\[ w_1 + v_1 w_{1,1} + v_2 w_{1,2} - v(w_{1,1} + w_{2,2}) = s_1 d_{1,1} + s_2 d_{2,2} + 2s_{1,2} d_{1,2}, \] (116)

where \( v \) is a material parameter with units of length squared over time, or kinematic viscosity. Other forms of non-local hardening can be treated similarly. The principal effect of non-local hardening is to introduce an additional length scale into the formulation, the expectation being that the effect on stability is not sensitive to the precise form of the non-local terms. Within the boundary layer, the scaling argument given in Section 2.2 reduces (116) to

\[ \varepsilon^\mu (w_{1,1} + v_1 w_{1,1} + v_2 w_{1,2}) - \varepsilon^{\mu-2} \mathcal{G} w_{1,2} = \varepsilon^{\mu-1} \mathcal{G} s_{1,2} v_1 w_{1,2}, \] (117)

where \( \mu \) is an as yet undetermined constant. Equations (23), (29), and (30) yield the hardening equation

\[ \mathcal{G} = \frac{m+\bar{n}+\tilde{\mu}}{m}. \]

It is apparent from (117) that the convective terms, which have characteristic exponents of the same order of magnitude as those in the normalization (60), give rise to an effective conduction-dominated stability criterion: the condition

\[ m + \bar{n} > 0. \]

Similarity solutions (119) then reduce (117) to

\[ \varepsilon = \varepsilon_0 \exp(-2\mathcal{G} \int \frac{m+\bar{n}}{m} \, ds), \]

which replaces (56) and (60) in Sections 4.1 and 4.2.

Evidently, the equation from 1 to 2, which has the same form as in Sections 4.1 and 4.2, and the exponents defined as in Sections 4.1 and 4.2, are sufficient for determining the stability of the conduction-dominated case, stability being.

This condition in Sections 4.1 and 4.2 is derived from the fact that hardening by itself has no effect on stability. However, it is interesting to note that the adiabatic stability condition for the conduction-dominated case, stability being...
where \( \mu \) is an as yet unknown exponent, and use has been made of normalization \((23), (29), \) and of definitions \((34), (70) \). The presence of the non-local term in the hardening equations introduces the additional dimensionless number

\[
\mathcal{Q} \equiv \frac{\nabla}{UL}.
\]  

(118)

It is apparent from \((117)\) that the non-local hardening term dominates over the convective terms, which can therefore be neglected to leading order. Balancing the exponents of the remaining terms yields \( \mu = \alpha + 1 \) in lieu of \((36) \). Finally, using normalization \((60) \) the reduced hardening equation can be expressed as

\[
- \mathcal{Q} \frac{\partial^2 \tilde{\psi}}{\partial \tilde{x}_2^2} = \mathcal{P}_{12} \frac{\partial \tilde{\psi}_1}{\partial \tilde{x}_2}.
\]  

(119)

Similarity solutions are now sought in the form \((86c), (85) \). The hardening equation \((119) \) then reduces to the ordinary differential equation

\[
- 2h'' = \mathcal{P} \tau |f''|.
\]  

(120)

which replaces \((91) \).

Evidently, the introduction of non-local hardening raises the order of the hardening equation from 1 to 2. This is precisely the effect which heat conduction has on the energy equation. Guided by this analogy, we assume the asymptotic form of \( h \) to be

\[
h \sim h_0 + h_1 \tilde{\tau}^\alpha,
\]  

(121)

which has the same structure as \((113) \). The asymptotic form of \( f, g \) and \( \tau \) postulated in Sections 4.1 and 4.2 remain valid here. The various constants and characteristic exponents defining the asymptotic structure of the boundary layer can be determined as in Sections 4.1 and 4.2. The stability condition is \( \varepsilon_f > 1 \). In the convection-dominated case, stability requires

\[
m + \tilde{l} > 0.
\]  

(122)

This condition implies that, in the presence of non-local hardening, the strain softening exponent has no effect on stability. Comparison of \((122) \) with \((112) \) suggests that non-local hardening compensates for—and eliminates—the destabilizing effect of mechanical softening. This is in analogy to the stabilizing effect of heat conduction on thermal softening. In the conduction-dominated case, the asymptotic analysis delivers a constant value of \( \varepsilon_f = 2 \), and the material is always stable.

A few fine points merit further discussion. Firstly, it is noteworthy that non-local hardening is not by itself sufficient to prevent the collapse of boundary layers. Indeed, it follows from \((122) \) that, in the absence of heat conduction, instabilities can be triggered by strong thermal softening, i.e. by a sufficiently negative value of \( \tilde{l} \). Secondly, it is interesting to note that, by eliminating the hardening exponent \( n \) from the adiabatic stability condition \((112) \), non-local hardening has a destabilizing effect when \( n > 0 \). Thus, in the absence of heat conduction, a material such that \( m + n + \tilde{l} > 0 \) but \( m + \tilde{l} < 0 \) is stable if its hardening is local but unstable if its hardening is non-local. Finally, in cases in which the characteristic length introduced by non-local hardening
is small, it follows that $s \ll 1$ and a "complex" boundary layer of the type described in Section 2.3 arises. Thus, while non-local hardening always dominates asymptotically close to the contact, this dominance may be confined to an exceedingly narrow band. In the remainder of the boundary layer, nonlocal effects are negligible and the convective terms in (117) dominate.

4.4. Vortex sheets as generalized solutions

In the preceding sections, vortex sheets have arisen as degenerate limits of steady boundary layers. These limiting solutions are approached as the stability limit of the material is attained. It is a point of some theoretical interest to demonstrate that vortex sheets are indeed generalized solutions, i.e. solutions in the sense of distributions, of the adiabatic boundary layer equations when $m + n + l = 0$. For simplicity, we consider the case of a rigid contact for which the boundary conditions at $\xi = 0$ reduce to $v_1 = v_2 = 0$. We begin by postulating the collapsed velocity profile

$$f'(\zeta) = H(\zeta),$$

(123)

where $H(x) = (x + |x|)/2$ is the Heaviside step function. It then follows that

$$f(\zeta) = \zeta \quad \text{for } \zeta \geq 0$$

(124)

and

$$f''(\zeta) = \delta(\zeta),$$

(125)

where $\delta$ is the Dirac delta function. Evidently, these expressions are compatible with the velocity boundary conditions at the contact. Next, rewrite (90) in the form

$$-af'' = \frac{1}{\mathcal{R}} \tau',$$

(126)

and require that it be satisfied in a distributional sense. Inserting (123)–(125) into (126), multiplying both sides by an arbitrary test function $\phi(\zeta)$ and integrating with respect to $\zeta$ from 0 to infinity leads to the identity

$$\int_0^\infty \tau'(\zeta) \phi(\zeta) \, d\zeta = -a \int_0^\infty \zeta \delta(\zeta) \phi(\zeta) \, d\zeta = 0.$$  \hspace{1cm} (127)

Since the test function $\phi$ is arbitrary, it follows that $\tau(\zeta) \equiv \tau_0 = \text{const}$ for $\zeta \geq 0$. The requirement that the hardening equation (91) be satisfied in a distributional sense leads to the integral statement

$$-a \int_0^\infty \zeta h'(\zeta) \phi(\zeta) \, d\zeta = \mathcal{P} \tau_0 \phi(0),$$

(128)

which, by virtue of the arbitrariness of $\phi$, requires that

$$h(\zeta) = \frac{\mathcal{P} \tau_0}{a} \delta(\zeta)$$

(129)

(provided that $\tau_0 \neq 0$). Thus, all the hardening occurs within the band of vanishing
Dynamic boundary layers and shear bands

thickness to which shearing is confined. Similarly, the energy equation (94) leads to the statement

\[ -a \int_0^\infty f(\zeta)g^e(\zeta)g'(\zeta)\phi(\zeta) \, d\zeta = \mathcal{F}_{\tau_0} \phi(0), \quad (130) \]

which is satisfied by setting

\[ g(\zeta) = \left[ \frac{\mathcal{F}_{\tau_0}}{a} \delta(\zeta) \right]^{1/(1+q)}. \quad (131) \]

Thus, all thermal effects are confined to the vortex sheet. Finally, we insert (125), (129) and (131) into the constitutive relation (88) and proceed to enforce it in a distributional sense. Since (88) applies only if the rate of deformation is non-zero, we restrict the range of integration to the interval \([0-0^+\]), with the result

\[ \tau_0 \int_0^{0^+} \phi(\zeta) \, d\zeta = 0 = \mathcal{F}_{\tau_0} \left( \frac{\tau_0}{a} \right)^{1/(1+q)} \int_0^{0^+} [\delta(\zeta)]^{m+n+1} \phi(\zeta) \, d\zeta. \quad (132) \]

This relation is identically satisfied provided that \(m+n+1\). Consequently, in this limit, the boundary layer equations admit vortex sheets as generalized solutions.

5. TRANSIENT BOUNDARY LAYERS

In this section we generalize the similarity techniques developed in the foregoing to account for transient effects. In particular, we construct a class of unsteady boundary layer solutions which characterizes the mechanical and thermal fields attendant to dynamically propagating shear bands such as develop in impact tests. The transient character of the solutions sought necessitates consideration of the full boundary layer equations, with rate terms included. The steady solutions derived in the preceding sections demonstrate that the effect of heat conduction is typically restricted to an exceedingly narrow region. Consequently, we confine our attention to the convection-dominated case in the interest of simplicity.

The governing equations are, therefore, (61)–(65) and (68). Solutions to these equations can be obtained by introducing the similarity variables

\[ \zeta = \tilde{x}_1 \tilde{x}_1^{-a}, \quad \tilde{\zeta} = \tilde{x}_1^{-1}, \quad (133) \]

and adopting the representation

\[ \tilde{\psi} = \tilde{x}_1^m f(\zeta, \tilde{\zeta}), \quad \tilde{\theta} = \tilde{x}_1^m g(\zeta, \tilde{\zeta}), \quad \tilde{\psi} = \tilde{x}_1 h(\zeta, \tilde{\zeta}), \quad (134) \]

which generalizes (86). Inserting representation (134) into the boundary layer equations and balancing powers of \(\tilde{x}_1\) yields the system of partial differential equations

\[ (1 - \tilde{\zeta} f_{,\tilde{\zeta}}) f_{,\tilde{\zeta}} + \tilde{\zeta} f_{,\tilde{\zeta}} f_{,\tilde{\zeta}} - a f_{,\tilde{\zeta}} = \frac{1}{\mathcal{A}} \tau \left( m \frac{f_{,\tilde{\zeta}}}{f_{,\tilde{\zeta}}^2} + \frac{h_{,\tilde{\zeta}}^2}{h_{,\tilde{\zeta}}} + \frac{g_{,\tilde{\zeta}}}{g} \right). \quad (135) \]
\[ g^s[(1 - \xi f_\xi)g_\xi + (-af + \xi f_\xi)g_\xi] = \mathcal{P} \tau |f_{\xi\xi}|, \]  
\[ (1 - \xi f_\xi)h_\xi + (-af + \xi f_\xi)h_\xi = \mathcal{P} \tau |f_{\xi\xi}|, \]  
\[ \tau = |f_{\xi\xi}|^m g^h h^n, \]  
\[ \]  
Eq. (136) 
Eq. (137) 
Eq. (138) 

Together with exponents (95). Remarkably, the similarity of the velocity, temperature and plastic work fields is preserved in the transient case. Evidently, (135), (136) and (137) reduce to their steady counterparts when all derivatives with respect to \( \xi \) vanish. To the boundary conditions formulated in Section 3 now we need to append suitable initial conditions at \( t = 0 \), or, equivalently, at \( \xi = 0 \). The initial state of the plate as it is struck by a flat-ended impactor is described by the fields 
\[ f_\xi(\xi, 0) = H(-\xi), \quad g(\xi, 0) = 1, \quad h(\xi, 0) = 1, \]  
\[ \]  
Eq. (139) 
where \( H \) is the Heaviside step function. Evidently, the initial conditions (139b) and (139c) are compatible with the boundary conditions (96) and (97) at infinity.

5.1. The transition to the steady state

We begin by investigating the evolution of the initial fields (139) towards the steady solutions treated in Section 3. We therefore suppose that the material is stable in the sense of (112) and, consequently, capable of sustaining steady boundary layers. Using \((1 - \xi f_\xi)\) as an integrating factor, (135), (136) and (137) can be recast in the form

\[ \left( \frac{f_\xi}{1 - \xi f_\xi} \right)_\xi = \frac{\mathcal{R}^{-1} s + a f_\xi}{(1 - \xi f_\xi)^2}, \]  
\[ \]  
Eq. (140) 
\[ g_\xi = \frac{g^s \mathcal{P} - s f_{\xi\xi} + (af - \xi f_\xi)g_\xi}{1 - \xi f_\xi}, \]  
\[ \]  
Eq. (141) 
\[ h_\xi = \frac{\mathcal{P} s f_{\xi\xi} + (af - \xi f_\xi)h_\xi}{1 - \xi f_\xi}, \]  
\[ \]  
Eq. (142) 
\[ s = |f_{\xi\xi}|^m g^h h^n \text{sgn}(f_{\xi\xi}). \]  
\[ \]  
Eq. (143) 

These equations are suggestive of the manner in which the transient solutions approach a steady state. Since \( 0 < f_\xi < 1 \), for every \( \xi \) there is a value of \( \xi \), denoted \( \xi^*(\xi) \), such that

\[ 1 - \xi^*(\xi)f^s(\xi, \xi^*(\xi)) = 0. \]  
\[ \]  
Eq. (144) 

At precisely this value of \( \xi \), the denominators on both sides of (140) vanish. For the equation to remain well-behaved the numerators must vanish as well, which necessitates \( f_\xi = 0 \) and the satisfaction of (90). Consequently, \( f(\xi, \xi) \) must coincide with the steady value of \( f \) at \( \xi \) for \( \xi \geq \xi^*(\xi) \). Setting \( f_\xi = 0 \) in (141) and (142), the same argument suggests that \( g(\xi, \xi) \) and \( h(\xi, \xi) \) coincide with the steady values of \( g \) and \( h \) at \( \xi \) for \( \xi \geq \xi^*(\xi) \). It therefore follows that the steady solution is attained in finite—albeit position-dependent—time. Indeed, the time required for the attainment of the steady state is

which, evidently, is determined by recurrent computations. Figure 6 shows the evolution of the steady and transient states, with the solid line indicating the steady state. Consequently, plastic deformations are attained at a critical state.

5.2. Propagating fronts

The similar considerations investigating the propagation of a shock wave in a material initially at stress \( n \), followed by a crystallographic transformation, are to occur at a critical state.

Conveniently, the constant \( n \) reduces simply to the initial state. At any given time, the solution of (130) can be regarded as an "analytical" solution, which follows by
Dynamic boundary layers and shear bands

![Graph showing evolution of the steady-state boundary in a plate struck by a flat-ended rigid impactor.]

Fig. 6. Evolution of the steady-state boundary in a plate struck by a flat-ended rigid impactor.

\[
\bar{P}^* = \bar{x}_1 \xi^*(\bar{x}_2 \bar{x}_1^{-a}),
\]

which, evidently, depends on position. The dependence of \(\bar{P}^*\) on \(\bar{x}_1^{-1}\) can be conveniently determined by recognizing that \(f_\xi(\xi, \xi^*(\xi))\) in (144) coincides with the steady state of \(f_\xi\) at \(\xi\). Figure 6 shows the evolution in the \(\bar{x}_1 - \bar{x}_2\) plane of the boundary between the steady and transient regions for the copper plate example treated in Section 3.1. As expected, the steady state is attained first on the impact surface and subsequently spreads into the interior of the plate. Interestingly, it follows from (144) that steady conditions on vertical planes \(\bar{x}_1 = \text{constant}\) are first attained for \(\bar{x}_2 \to -\infty\) at \(\xi = 1\), i.e. at time \(\bar{t} = \bar{x}_1\). But this is precisely the time of arrival of the rigid impactor. Consequently, points at depths greater than the total depth of indentation never attain a steady state within the duration of the test.

5.2. Propagating shear bands and shear band tip speeds

The similar transient equations (135)–(138) provide a convenient framework for investigating the transition from stable to unstable boundary layers, leading to the formation of a shear band. As a simple model of this transition, we assume that the material initially exhibits stable behavior characterized by a high hardening exponent \(n\), followed by an unstable regime of low—possibly negative—\(n\). The stress varies continuously across this transition. The reduction in the rate of hardening is assumed to occur at a critical accumulated plastic work \(\bar{\tilde{w}}_c\), i.e. when

\[
\bar{\tilde{w}} = \bar{\tilde{w}}_c.
\]

Conveniently, by virtue of the similarity of the transient solutions this condition reduces simply to \(\bar{\tilde{w}}(\xi, \tilde{\xi}) = \bar{\tilde{w}}_c\), and all fields retain similarity through the transition.

At any given time, the locus of points at which the material is critical is defined parametrically by the equation

\[
\tilde{w}(\bar{x}_2 \bar{x}_1^{-a}, \bar{t} \bar{x}_1^{-1}) = \bar{\tilde{w}}_c.
\]

This curve encloses a region of highly deformed unstable material which may therefore be regarded as a shear band. The position of the tip of this region, or “shear band tip”, follows by particularizing (147) to \(\bar{x}_2 = 0\), which yields the condition
\[ \tilde{\omega}(0, \tilde{\xi}_1^{-1}) = \tilde{\omega}_e. \]  

(148)

This in turn requires \( \tilde{\xi}_1^{-1} = \xi_c = \text{constant} \) and, consequently, the shear band tip proceeds at the constant speed
\[ \tilde{\nu} = \frac{1}{\xi_c}. \]  

(149)

It bears emphasis that the constancy of the shear band tip speed is a direct consequence of the similarity of the solution and the form (133b) of the time-like similarity variable \( \xi \), which represents a reciprocal speed. In view of (149), the shear band tip speed can be computed simply by integrating the transient similar equations until such time \( \xi_c \) as the critical plastic work \( \tilde{\omega}_c \) is attained at \( \xi = 0 \). It should be noted that the shear band tip speed does not depend on the form of the constitutive relation in the unstable regime.

5.3. Example

By way of example we consider a copper plate such as treated in Section 3.1, with hardening exponents \( n = 0.1 \) and \( n = 0 \) in the stable and unstable regimes, respectively. We verify that, by taking all remaining parameters for copper as in Table 1, \( n = 0.1 \) gives \( m + n + l = 0.04 > 0 \), while \( n = 0 \) yields \( m + n + l = -0.06 < 0 \). The normalized critical plastic work \( \tilde{\omega}_c \) is set to 5. It should be carefully noted that the choice \( n = 0 \) of unstable hardening exponent, while influencing the postcritical behavior, has no bearing on the solution prior to the attainment of the critical condition. In particular, the shear band tip speed is independent of the choice of unstable hardening exponent. The smooth initial velocity profile
\[ f_\xi(\xi, 0) = \frac{1}{2} \left[ 1 - \tanh \left( \frac{\xi}{\xi_c} \right) \right], \]  

(150)

in which \( \xi_c \) represents a small width parameter, is adopted in lieu of (139a) for numerical purposes. Equations (140), (141) and (142) provide a convenient basis for numerical calculations. The derivative with respect to \( \xi \) in the left hand side of (140) can be eliminated by direct integration, with the result
\[ f_\xi = (1 - \xi f_\xi) \left[ \frac{\mathcal{R}^{-1} s}{(1 - \xi f_\xi)^2} - \frac{\mathcal{R}^{-1} s(0, \xi)}{(1 - \xi f_\xi(0, \xi))^2} \right] \]
\[ + \int_0^{\xi_c} \frac{\xi f_\xi [af(1 - \xi f_\xi) - 2 \xi s \mathcal{R}^{-1}]}{(1 - \xi f_\xi)^3} d\xi. \]  

(151)

In this expression we have effected an integration by parts on \( s \), which has the beneficial effect of lowering by one the order of the derivatives of \( f \) which need to be computed. Equations (141), (142) and (151) directly give the rates of \( f, g \) and \( h \) and can be integrated in \( \xi \) simply by the forward-Euler method.

Figure 7 shows dependence of the Reynolds number on the initial shear band width \( \xi_c \), of the initial temperature \( \tilde{T}_v \), and on the initial heat flux \( \tilde{q} \). Figure 8 displays the temperature and velocity profiles for the unstressed sheet, \( \tilde{T}_v = 0.25 \). It is seen that the temperature and velocity profiles are highly wrinkled, indicating a small thickness of the sheet, which is to be expected from the solution.
Local consequence variable and tip speed can until such time \( \zeta \) that the shear \( \alpha \) in the unstable

\begin{equation}
(149)
\end{equation}

Figure 7 shows the dependence of the normalized shear band tip speed on the Reynolds number and the critical plastic work. It should be carefully noted that the shear band tip speeds depicted in Fig. 7 are normalized by the impact velocity. The width \( \zeta \) of the initial velocity profile is set to 0.04. The shear band tip speed is seen to be greatly in excess of the impact velocity and to increase sharply with the Reynolds number at low impact velocities, in agreement with the observations of Zhou et al. (1995). At sufficiently high impact velocities, the normalized shear band tip speed saturates and appears to tend asymptotically to a constant value. As expected, high values of \( \tilde{w}_c \) have the effect of retarding the propagation of the band.

Figure 8 displays the dependence of the shear band tip speed on the initial band width parameter \( \zeta \), the rate sensitivity exponent \( m \), the stable hardening exponent \( n \) and the thermal softening exponent \( l \), for a range of Reynolds numbers. In these calculations the critical plastic work \( \tilde{w}_c \) is set to 5. The initial band width parameter \( \zeta \) furnishes a simple vehicle for assessing the sensitivity of the shear band tip speed to the initial conditions. It is evident from Fig. 8 that this sensitivity is very high. In actual tests, a certain degree of scatter in the initial conditions is often unavoidable. The early time conditions are also influenced by such geometrical features as the size of the notch root radius in pre-notched specimens. Therefore, a good deal of scatter in shear band tip speed measurements is to be expected. By comparison, the dependence of the shear band tip speed on the exponents \( m \), \( n \) and \( l \) is relatively weak.

The time evolution of the velocity, plastic work, temperature and rate of deformation fields is displayed in Figs 9, 10, 11 and 12, respectively. In these calculations the Reynolds number is set to 10, the width parameter \( \zeta \), to 0.1, and the critical plastic work \( \tilde{w}_c \) to 5. The remaining dimensionless numbers of the flow are \( \mathcal{P} = 5 \) and \( \mathcal{F}_v = 0.25 \). It should be noted that in all plots the \( \tilde{x}_2 \)-axis scale has been magnified by a factor of 2 to aid visualization. The figures clearly bring forth the fully two-dimensional and time-dependent nature of the solution. Figure 9 illustrates how the \( \tilde{v}_1 \)-profile spreads out in time from the nearly stepwise initial conditions. As expected from the scaling laws developed in Section 2.2, the \( \tilde{v}_2 \) component of velocity is negligibly small relative to \( \tilde{v}_1 \).

\begin{equation}
(150)
\end{equation}

\begin{equation}
(151)
\end{equation}
With increasing time, the level contours of plastic work appear to emanate from the origin and to broaden as they shoot downstream, Fig. 10. This type of growth has been observed by Needleman (1989) and by Chou et al. (1992) in finite element simulations of dynamic shear banding. The plastic work at \( \zeta = 0 \) attains its critical value \( \bar{\bar{w}}_c \) at \( \xi_c = 0.113 \), resulting in a normalized shear band tip speed \( \bar{V} = 1/\xi_c = 8.84 \). The boundary of the shear band in Fig. 10 coincides with the level contour \( \bar{w} = \bar{w}_c = 5 \). As may be seen, the shear band emanates from the origin at \( \bar{\tau} = 0 \) and subsequently propagates downstream.

It is also noteworthy that the 100 \( \mu s \) time scale of 229°C.

The rate of this feature is not constant over the entire period.
propagates downstream at the theoretical speed. The remarkable thinness of the band is also noteworthy. The temperature field exhibits a similar evolution, Fig. 11. Over the 100 μs time interval displayed in the figure, the temperatures reach a peak value of 229°C.

The rate of deformation field differs from the plastic work and temperature fields in that it is not similar but has the representation

$$\hat{d}_{12} = \hat{x}_1^{-a} f_{\xi_1}(\zeta, \xi).$$  \hspace{1cm} (152)

This feature confers the field a more complex spatial structure and temporal evolution. For instance, the levels of the field on $\tilde{x}_2 = 0$ no longer travel downstream at constant speed, as in the case of the plastic work and temperature fields. Instead, at fixed $\tilde{x}_1$, the rate of deformation is clearly non-monotonic in time: it exhibits a rapid initial increase followed by a gradual decrease, seemingly leading to a steady state in which
the collapse of the shear band is checked by the stabilizing effect of the surrounding stable material. The profile of the rate of deformation field is shown in Fig. 13. The box function demarcates the unstable region. The bipartite structure of the rate of deformation profile is clearly apparent in the figure.

6. DISCUSSION

We have developed a boundary layer theory for thermoviscoplastic solids which facilitates the analytic characterization of certain types of flows. The defining property of these flows while being slow to permit cold hardening, the realistic description of the theory can be curved and ax when applied to the introductory dimensions by
of these flows is that the attendant fields exhibit a rapid variation in one direction while being slowly varying in the remaining directions. The theory is general enough to permit consideration of large plastic deformations, rate sensitivity, mechanical hardening, thermal softening, and heat convection and conduction, thus affording a realistic description of high-rate deformation processes. With minor modifications, the theory can be extended so as to encompass more general geometries, including curved and axisymmetric boundary layers. The theory is particularly advantageous when applied to problems which are tractable by similarity methods. In these cases, the introduction of a similarity variable effectively reduces the number of spatial dimensions by one. The implication of this reduction is that fully two-dimensional
solutions can be obtained with the same computational effort and sophistication as is required to obtain one-dimensional solutions.

We have systematically exploited this feature of the theory in pursuing a representative application, namely, the analysis of the internal boundary layer which develops in a plate which is struck by a flat-ended impactor at high velocities. Of particular interest here is to understand the mechanisms by which a shear band forms at the boundary and propagates into the plate. To render this question meaningful, an operative definition of shear band needs to be agreed upon first. In our analysis, we regard shear banding as the outcome of a constitutive instability which renders the material incapable of supporting steady boundary layers. This stability criterion returns Molinari further identify stability.

Remarkably, characterized by sinusoidal assumption in the small deformation deformations we consider, behavior is expected from the theory of speed which is smaller than that involving relativistic scenario. Using propagating at an impact speed at (1995) to remain consistent with the impact velocity.

Finally, the carefully noted that the crack tip may be treated as a singular field.
returns Molinari and Clifton’s (1982) localization condition in the adiabatic case. We further identify a shear band with the domain over which the material has lost stability.

Remarkably, dynamically propagating shear bands can be conveniently characterized by similarity methods for a restricted class of constitutive models. The key assumption in these models is that the material exhibits a high rate of hardening at small deformations which renders it stable, and a low rate of hardening at large deformations which causes it to lose stability. The transition from stable to unstable behavior is effected at a critical value of the accumulated plastic work. It then follows from the theory that the tip of the shear band propagates into the plate at a constant speed which is greatly in excess of the impact velocity. The experiments of Mason et al. (1994) and Zhou et al. (1995) on pre-notched C-300 maraging steel plates, while involving relatively low impact velocities ($\sim 30$ m s$^{-1}$), appear consistent with this scenario. Using optical techniques, they observed a feature akin to a shear band tip propagating at speeds as high as 1200 m s$^{-1}$. The tip speed was found by Zhou et al. (1995) to remain steady during the greater part of the test and to increase sharply with the impact velocity. These observations are in accordance with our calculations.

Finally, the essential differences between shear bands and crack tips should be carefully noted. The motion of a crack tip involves processes of separation which lie squarely outside the purview of local constitutive theories. Therefore, the description of crack tip motion requires mechanical postulates which are independent of the constitutive equations. In addition, crack tips carry along with them autonomous singular fields whose form is independent of the geometry of the body. By contrast,
shear banding, as understood here, is strictly a consequence of the constitution of the material. Because dynamic shear bands tend to be very elongated, their leading front may be regarded as a propagating “tip”. The entity thus defined is clearly identifiable by, for instance, optical methods (Mason et al., 1994; Zhou et al., 1995) and thus amenable to experimental observation. As all other features of a shear band, the speed of propagation of its tip is dictated by constitutive behavior. Our similarity solutions, and the finite element simulations of Needleman (1989), however, show a certain degree of broadening, or “diffusion”, in the developing shear band. In addition, the shear band tip certainly does not carry singular autonomous fields with it, but is merely a salient feature of an otherwise continuous field. These observations have led some authors to categorize dynamic shear band growth as diffusive and to discard the notion of a shear band tip altogether. However, provided that the physical nature of shear band tips is clearly understood, they can play a useful role, for instance, as a basis for comparisons between theory and experiment.

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