A VARIATIONAL BOUNDARY INTEGRAL METHOD FOR
THE ANALYSIS OF 3-D CRACKS OF ARBITRARY
GEOMETRY MODELED AS CONTINUOUS
DISTRIBUTIONS OF DISLOCATION LOOPS

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SUMMARY
A finite element methodology for analysing propagating cracks of arbitrary three-dimensional geometry is
developed. By representing the opening displacements of the crack as a distribution of dislocation loops and
minimizing the corresponding potential energy of the solid, the kernels of the governing integral equations
have mild singularities of the type 1/R. A simple quadrature scheme then suffices to compute all the element
arrays accurately. Because of the variational basis of the method, the resulting system of equations is
symmetric. By employing six-noded triangular elements and displacing midside nodes to quarter-point
positions, the opening profile near the front is endowed with the correct asymptotic behaviour. This enables
the direct computation of stress intensity factors from the opening displacements. The special but important
cases of periodic and semi-infinite cracks are addressed in some detail. Finally, the geometry of propagating
cracks is updated incrementally by recourse to a pseudodynamic crack-tip equation of motion. The crack is
continuously remeshed to accommodate the ensuing changes in geometry. The performance of the method is
assessed by means of selected numerical examples.

1. INTRODUCTION
The problem considered in this paper concerns the deformation and growth of a crack of
arbitrary three-dimensional geometry embedded in a homogeneous and infinite linear elastic
solid. Numerous boundary integral methods which reduce this class of problems to the solution
of integral equations defined on the faces of the crack are presently in existence.\(^1\)\(^-\)\(^9\) Most of these
methods, however, make use of the Somigliana identity as a means of effecting the reduction to
the boundary, which results in highly singular integral equations of difficult numerical treat-
ment.\(^2\)\(^-\)\(^4\)\(^,\)\(^5\)\(^,\)\(^7\) Instead, here we propose to represent the opening displacements of the crack as
continuous distributions of dislocation loops. The energy of the solid can then be computed from
known expressions for the interaction energy of a pair of dislocation loops.\(^10\) The geometry of the
loops and their Burgers vectors are readily related to the crack opening displacements, which are
then determined by minimization of the total potential energy of the solid. Because the governing
integral equations are derived from a variational principle, the systems of equations which result
upon discretization of the crack are symmetric. The method shares this desirable property with
the Galerkin boundary element formulation of Maier and Polizzotto.\(^11\) The distributed disloca-
tion loop representation gives rise to integral equations which are only mildly singular, which
greatly facilitates their numerical treatment. The present work extends the formulation of Bui,\(^1\)
based on double-layer potentials, and the variational formulation of Clifton and Abon-Sayed\(^3\)
and Clifton\(^6\) based on the fundamental solution for a dislocation segment. Both of these

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formulations were restricted to planar cracks and mode I loading. The governing equations derived by the dislocation superposition method can be brought into correspondence with those proposed by Weaver.\textsuperscript{12}

The distributed dislocation loop representation furnishes a convenient basis for the development of finite element schemes capable of analysing three-dimensional growing cracks. By employing six-noded triangular elements and displacing midside nodes to quarter-point positions, the opening profile near the front is endowed with the correct asymptotic behaviour. This enables the accurate computation of stress intensity factors from the opening displacements. We address in some detail the special but important cases of periodic and semi-infinite cracks. Cracks of this type arise naturally in micromechanical studies of fracture processes in materials with microstructure. We show that the analysis of periodic cracks can be restricted to one period, and that semi-infinite cracks need only be discretized over a finite region.

Crack growth is modelled by assigning a velocity to the crack front according to a pseudodynamic crack-tip equation of motion,\textsuperscript{13} and by enforcing a suitable kinking criterion such as the vanishing of the local mode II stress intensity factor.\textsuperscript{14–16} Other types of crack-tip equations of motion and kinking criteria, such as those adopted by Gao and Rice,\textsuperscript{17} Fares,\textsuperscript{18} Bower and Ortiz,\textsuperscript{19,20} and others, can be accommodated with equal ease. The analysis proceeds incrementally by a time-stepping procedure similar to that advocated by Fares.\textsuperscript{18} The mesh is continuously adapted to the changing geometry of the crack.

The paper is structured as follows. The variational foundations of the method and the distributed dislocation loop representation of the opening displacements are developed in Section 2. Matters of finite element implementation are taken up in Section 3, with special attention given to questions of singular integration of element arrays, the calculation of stress intensity factors, crack growth criteria and the handling of periodic and semi-infinite cracks. Selected test examples are collected in Section 4 which demonstrate the accuracy and versatility of the method. Overall, we find that satisfactory accuracy in the opening displacements and stress intensity factors is obtained with practical meshes. The ability of the method to march a crack through unstable regimes involving extensive crack arrest and coalescence is particularly noteworthy.

2. INTEGRAL EQUATION FORMULATION

Consider a three-dimensional crack of arbitrary geometry in an infinite elastic body subjected to general loading. Let the surface of the crack be \( S \) with contour \( C = \partial S \), and let \( \mathbf{u}(\mathbf{x}), \mathbf{x} \in S \), be the displacement jump across \( S \), with components \( U_i \) relative to a Cartesian basis \( e_i \). Our first aim is to compute the strain energy of the cracked solid. This is accomplished by representing the opening displacements as a continuous distribution of dislocation loops, and subsequently using known expressions for dislocation loop energies.

The dislocation distribution equivalent to \( \mathbf{u}(\mathbf{x}) \) is determined as follows. For \( i = 1, 2, 3 \), let \( C_{ik_i} \) be the level contours on \( S \) corresponding to values of \( U_i = K_i \Delta U_i \) (no sum), \( K_i = 0, 1, 2, \ldots, N_i \). This defines three sets of level contours inscribed on the crack surface, Figure 1. Since \( U_i = 0 \) at the crack front \( C \), it follows that \( C_{i0} = C \). We begin by replacing the original opening displacement field by one consisting of discrete steps of magnitude \( \Delta U_i \) across the level contours \( C_{ik_i} \). Evidently, the stepwise opening displacement field so defined approaches \( \mathbf{u}(\mathbf{x}) \) as \( \Delta U_i \to 0 \). Each contour \( C_{ik_i} \) may be viewed as carrying a dislocation of Burgers vector

\[
\mathbf{b}_i = \Delta U_i e_i \quad \text{(no sum on } i) \tag{1}
\]

Note that, by definition, the Burgers vector for the loop \( C_{ik_i} \) points in the \( i \)th co-ordinate direction and has a constant magnitude equal to \( \Delta U_i \). The elastic interaction energy between two
dislocation loops \( C_1 \) and \( C_2 \) of Burgers vectors \( b_1 \) and \( b_2 \) in an infinite linear elastic solid is given by\(^\text{10}\)

\[
W_{12} = \frac{\mu}{2\pi} \oint_{C_1} \oint_{C_2} \frac{(b_1 \cdot dl_2)(dl_2 \cdot dl_1)}{R} - \frac{\mu}{4\pi} \oint_{C_1} \oint_{C_2} \frac{(b_1 \cdot dl_1)(b_2 \cdot dl_2)}{R} \\
+ \frac{\mu}{4\pi(1-v)} \oint_{C_1} \oint_{C_2} (b_1 \times dl_1) \cdot T \cdot (b_2 \times dl_2)
\]

where \( dl_1 \) and \( dl_2 \) are vectors of infinitesimal length tangent to \( C_1 \) and \( C_2 \), respectively, \( \mu \) is the shear modulus, \( v \) Poisson’s ratio, \( R \) the length of the relative position vector between \( C_1 \) and \( C_2 \), and

\[
T_{ij} = \frac{\partial^2 R}{\partial x_i \partial x_j}
\]

Therefore, using the dislocation representation (see Figure 2) of the opening displacement and invoking the principle of superposition, the strain energy of the cracked solid follows as

\[
W[u] = \frac{1}{2} \lim_{\Delta U_1, \Delta U_2, \Delta U_3 \to 0} \sum_{i=1}^{3} \sum_{j=1}^{3} \sum_{k_i=0}^{N_i} \sum_{k_j=0}^{N_j} \frac{\mu}{2\pi} \oint_{C_{i,k_i}} \oint_{C_{j,k_j}} \frac{(b_i \cdot dl_{JK})(b_j \cdot dl_{JK})}{R} \\
+ \frac{\mu}{4\pi(1-v)} \oint_{C_{i,k_i}} \oint_{C_{j,k_j}} \frac{(b_i \times dl_{i,k_i}) \cdot T \cdot (b_j \times dl_{i,k_j})}{R}
\]

The factor 1/2 in this expression compensates for the fact that the double sum accounts for the interaction energy between each pair of loops twice. The computation of the strain energy is completed by passing to the limit of \( \Delta U_i \to 0 \). To this end, let \( \mathbf{m} \) be a unit surface vector (i.e., a unit vector tangent to \( S \)) normal to \( C_{i,k_i} \). Since \( C_{i,k_i} \) is a level contour of \( U_i \), it follows that

\[
\mathbf{m} = \nabla U_i/|\nabla U_i|
\]

Let \( t \) measure the distance from the loop along \( \mathbf{m} \), and let \( \Delta t \) be the distance along \( \mathbf{m} \) between the consecutive loops \( C_{i,k_i} \) and \( C_{i,(k_i+1)} \), Figure 1. The derivative of \( U_i \) with respect to \( t \) is given by

\[
\frac{dU_i}{dt} = \nabla U_i \cdot \mathbf{m} = |\nabla U_i|
\]
where use has been made of the identity (5). Hence, for small $\Delta U_i$, one has

$$\Delta U_i \approx |\nabla U_i| \Delta t$$

and the Burgers vectors (1) become

$$b_i \approx |\nabla U_i| \Delta t e_i$$

Furthermore, the line element along a contour may be expressed as

$$dl = d\mathbf{n} \times m = d\mathbf{n} \times \nabla U_i / |\nabla U_i|$$

where $ds$ is the element of arc length, and $\mathbf{n}$ is the unit normal vector to the surface. Substituting (8) and (9) into (4), and noting that in the limit of $\Delta U_i \to 0$, $\Delta t ds$ may be identified with the element of surface area $dS$, one finds

$$W[u] = \frac{\mu}{4\pi} \int_S \left[ \sum_{j=1}^{\infty} \frac{[e_{ij} \cdot (\mathbf{n} \times \nabla U_j)_1]}{R} [e_{ij} \cdot (\mathbf{n} \times \nabla U_j)_2] dS_1 dS_2 \right]$$

$$\quad - \frac{\mu}{8\pi} \int_S \sum_{j=1}^{\infty} \frac{[e_{ij} \cdot (\mathbf{n} \times \nabla U_j)_1]}{R} [e_{ij} \cdot (\mathbf{n} \times \nabla U_j)_2] dS_1 dS_2$$

$$\quad + \frac{\mu}{8\pi (1 - \nu)} \int_S \sum_{j=1}^{\infty} \frac{[(e_{ij} \cdot (\mathbf{n} \times \nabla U_j)_1)] \cdot T \cdot [(e_{ij} \times (\mathbf{n} \times \nabla U_j)_2)] dS_1 dS_2$$

where $(\cdot)_1$ and $(\cdot)_2$ denote two different points on $S$, and $R$ is the distance between these two points. Equation (10) is the sought expression for the strain energy of the cracked solid. It takes the form of a double integral over the crack surface $S$. An advantage of the present formulation which is immediately apparent from (10) is that the kernels of all integrals have singularities of the type $R^{-1}$, i.e. are only mildly singular. As shown in Section 3.2, this greatly facilitates their numerical treatment.

For purposes of numerical implementation, it proves convenient to recast (10) in terms of surface co-ordinates. To this end, let $S$ be parametrized by means of surface co-ordinates $\theta^a (a = 1, 2)$ with $a^a$ and $a^a$ denoting the associated contravariant and covariant basis vectors, respectively, and $a_{\alpha\beta} = a_\alpha \cdot a_\beta$ the corresponding metric tensor. Identify $a^3$ and $a^3$ with the unit normal vector to the surface. Denote by $u^i$ and $u_i$ the components of $u$ relative to the surface bases $a_i$ and $a^i$, respectively. Expressing $\nabla U$ in terms of surface components, and after some trite manipulations (vide Appendix), (10) may be rewritten in the form

$$W[u] = \frac{\mu}{4\pi} \int_S \int_S \frac{[(a_{i1})_1 \cdot (u^j_2 \epsilon^{j\alpha} a_\alpha)_2]}{R} \frac{[(a_{j2})_2 \cdot (u^i_1 \epsilon^{i\alpha} a_\alpha)_1]}{dS_1 dS_2}$$
\[ -\frac{\mu}{8\pi} \int_S \int_S \frac{(u|_\varepsilon, \varepsilon^a) (u|_\varepsilon, \varepsilon^a)}{R} dS_1 dS_2 + \frac{\mu}{8\pi(1-v)} \int_S \int_S (u^a|_\varepsilon, \varepsilon^a) \cdot \mathbf{T} \cdot (u^b|_\varepsilon, \varepsilon^b) dS_1 dS_2 \] (11)

In this expression, we have adopted the usual convention of using upper (lower) indices to denote contravariant (covariant) components. In addition, \( \varepsilon^{11} = \varepsilon^{22} = 0 \) and \( \varepsilon^{12} = -\varepsilon^{21} = 1/\sqrt{a}, a = a_{11}a_{22} - a_{12}a_{21} \), are the contravariant components of the permutation tensor, and \( u^a|_\varepsilon \) and \( u|_\varepsilon \) are components of the covariant derivatives of \( u \). A definition of the covariant derivative is given in the Appendix.

The potential energy of the solid follows as the sum of the strain energy (10) and the potential energy of the applied loads. Without loss of generality, we shall confine our attention to the case in which tractions \( T_i \) are applied directly to the faces of the crack. Then, the potential energy of the solid is

\[ \Phi[u] = W[u] - \int_S \mathbf{T} \cdot \mathbf{u} dS \] (12)

For given tractions, the opening displacements of the crack may be obtained by minimizing the potential energy \( \Phi \). The distribution of stress intensity factors over the crack front \( C \) can then be extracted from the known opening displacements. As is evident from (10), \( \Phi \) defines a positive-definite quadratic form in \( u \). Consequently, Rayleigh–Ritz methods of approximation based on the constrained minimization of \( \Phi \) over a finite dimensional interpolation space result in symmetric systems of equations. One such method based on a finite element discretization of the crack surface is developed in Section 3.

3. NUMERICAL PROCEDURE

In this section, we develop a numerical methodology for the analysis of growing cracks of arbitrary three-dimensional geometry. Each successive configuration of the crack is spatially discretized into six-noded triangular finite elements, and the opening displacements are interpolated from their nodal values by means of conventional shape functions. The nodal opening displacements are then determined by minimization of the potential energy (12). The calculation of element arrays is facilitated by the low-order singularity of the kernels resulting from the distributed dislocation representation. In addition, six-noded triangular elements furnish a convenient means of ensuring the correct asymptotic form of the opening displacements near the front, by the simple device of displacing midside nodes to quarter-point locations. Then, the stress intensity factors may be extracted from the displacement solution by comparing the opening profile near the front with its known asymptotic form. The stress intensity factors thus computed are then inserted into a suitable crack-tip equation of motion which determines the rate of growth of the crack. The finite element mesh is continuously adapted to accommodate the growth of the crack. Semi-infinite and periodic cracks arise frequently in applications and require special treatment. We show that the analysis of periodic cracks can be conveniently restricted to one period and that semi-infinite cracks need only be discretized over a finite region.

3.1. Finite element discretization

We begin by approximating the surface \( S \) by a polyhedral surface \( S_h = \bigcup_{i=1}^{N_h} \Omega^i \) composed of \( N_h \) triangular \( \Omega^i \) facets or ‘elements’. Here, the symbol \( h \) alludes to some measure of the mesh size.
Each facet is referred to a local Cartesian frame \{s_1, s_2, s_3\}, with \{s_1, s_2\} spanning the plane of the element and the s_3-axis pointing in the normal direction. Because of the flatness of the elements, covariant derivatives reduce to ordinary partial derivatives and (11) simplifies to

\[
W[u] = \left[ \frac{\mu}{4\pi} \int_{S_1} \int_{S_2} \frac{[(a_1) \cdot (u_{1,3} e_x \delta_3) + (a_2) \cdot (u_{1,3} e_y \delta_3)]}{R} \, ds_2 \, ds_1 \right] \\
- \frac{\mu}{8\pi} \int_{S_1} \int_{S_2} \frac{(u_{3,3} e_x \delta_3) + (u_{3,3} e_y \delta_3)}{R} \, ds_1 \, ds_2 \\
+ \frac{\mu}{8\pi(1-v)} \int_{S_1} \int_{S_2} (u_{3,3} e_x \delta_3 - u_{3,3} e_y \delta_3) \cdot (u_{3,3} \delta_3) \, ds_1 \, ds_2 \\
\equiv \int_{S_1} \int_{S_2} G_{ij}(x_1, x_2) u_{i,j}(x_1) u_{j,i}(x_2) \, ds_1 \, ds_2
\]  

(13)

where \( G_{ij} \) represents the ‘kernel’ of the governing integral equation.

Next, we introduce a finite element discretization of the opening displacements in the usual manner. The use of triangular elements enables meshes to be constructed simply by triangulation, which in turn facilitates the remeshing of growing cracks. Six-noded elements are especially attractive on account of the fact that, by displacing the midside nodes to quarter-point locations, a parabolic opening profile is obtained near the crack front. The discretized opening displacement and displacement gradient fields take the form

\[
u_i = \sum_{c=1}^{N_v} \sum_{a=1}^{6} u_{ia} N_{a,c}^v(s_1, s_2), \quad u_{i,a} = \sum_{c=1}^{N_v} \sum_{a=1}^{6} u_{ia} N_{a,c}^v(s_1, s_2)
\]  

(14)

where \( u_i \) are the nodal opening displacements and \( N_a^v \) the element isoparametric shape functions.\(^{21}\) We shall denote by \( u \) the array of nodal displacements \( \{u_1, \ldots, u_v\} \).

Substituting (14) into the potential energy (12) and making use of (13), the discretized potential energy of the solid reduces to

\[
\Phi_h = \frac{1}{2} \sum_{a=1}^{N} \sum_{b=1}^{N} K_{i,a,b} u_{i,b} u_{j,a} - \frac{1}{2} \sum_{a=1}^{N} f_{ia} u_{i,a}
\]  

(15)

where \( N \) is the total number of nodes and the ‘stiffness’ matrix \( K_h \) and effective force vector \( f_h \) are assembled from element contributions as

\[
K_{i,a,b} = \sum_{c=1}^{N_v} \sum_{a=1}^{6} \int_{\Omega} \int_{\Omega'} G_{ij}(x_1, x_2) N_{a,c}^v(x_1) N_{b,a}^v(x_2) \, d\Omega_1 \, d\Omega_2
\]

\[
f_{ia} = \sum_{c=1}^{N_v} \int_{\Omega'} T_c N_{a,c}^v \, d\Omega
\]

(16)

Because of the symmetries inherent to the kernel \( G_{ij} \), the stiffness matrix \( K \) is automatically symmetric, i.e. \( K_{i,a,b} = K_{b,a,i} \). The present method shares this desirable attribute with the Galerkin boundary element formulation of Maier and Polizzotto.\(^{11}\) It should also be noted that, due to the integral character of the governing equations, the assembly of the stiffness matrix here requires a double loop over all elements in the mesh. The nodal displacements can be determined \( \text{à la} \) Rayleigh–Ritz by minimizing the discretized potential energy (15), which yields the system of linear equations

\[
\sum_{b=1}^{N} K_{i,a,b} u_{j,b} = f_{i,a}
\]  

(17)
For known nodal forces \( f_k \) and a given crack geometry \( S \), equation (17) determines the nodal opening displacements \( u_k \), the expectation being that the discretized opening displacement field will converge to the exact solution as \( h \to 0 \), i.e. as the mesh size is allowed to become vanishingly small. A convergence study of the method is pursued in Section 4.1. In applications involving growing cracks, the surface \( S \) varies continuously and (17) applies to each successive configurations adopted by the crack.

3.2. Singular integration

Next we address the numerical computation of the integrals (16) defining the stiffness matrix \( K_h \). Integrals involving pairs of distinct elements, \( e_1 \neq e_2 \), pose no special difficulties and can be evaluated by conventional quadrature rules. The troublesome terms are those for which \( e_1 = e_2 \), representing the self-energy of each element in the mesh. In this case the functions to be integrated become unbounded and care must be exercised in order to properly evaluate the integrals. Here we develop a variation of the standard method of 'extraction of the singularity', whereby the singular integrals are separated into a bounded part, which can be handled by the usual quadrature rules, and a singular part which can be integrated explicitly.

We concern ourselves with integrals of the type

\[
K^s_{ij} = \int_{\Omega'} \int_{\Omega'} G_{ij}^s(x, x') N_{a, x}(x) N_{b, x}(x') \, d\Omega \, d\Omega'
\]  

(18)

Since

\[
G_{ij}^s(x, x') \sim 1/|x - x'| \quad \text{as} \quad r = |x - x'| \to 0
\]

(19)

the function defined by

\[
\phi_{ij}^s(x, \theta) = \lim_{r \to 0} r G_{ij}^s(x, x') N_{a, x}(x) N_{b, x}(x')
\]

(20)

is bounded. Here \( \theta \) is the angle made by the relative position vector \( x' - x \) and, say, the co-ordinate direction \( s_1 \). To isolate the singularity in the integrand, we recast (18) as

\[
K^s_{ij} = \int_{\Omega'} \int_{\Omega'} \left[ G_{ij}^s(x, x') N_{a, x}(x) N_{b, x}(x') - \frac{\phi_{ij}^s(x, \theta)}{|x - x'|} \right] \, d\Omega \, d\Omega'
\]

\[+ \int_{\Omega'} \int_{\Omega'} \frac{\phi_{ij}^s(x, \theta)}{|x - x'|} \, d\Omega \, d\Omega' = I + II
\]

(21)

The first integral is bounded and can be computed by a double quadrature over the domain of the element. This gives

\[
I = \sum_p \sum_q w_p^s w_q^s \left[ G_{ij}^s(\xi_p, \xi_q) N_{a, x}(\xi_p) N_{b, x}(\xi_q) - \frac{\phi_{ij}^s(\xi_p, \theta_p)}{|\xi_p - \xi_q|} \right]
\]

(22)

where \( \xi_p \) and \( w_p^s \) are the quadrature points and weights of element \( \Omega^s \), respectively, \( \theta_p \) denotes the angle subtended by \( \xi_p - \xi_q \) with the \( s_1 \)-axis, and the sums extend over all quadrature points. All terms in (22) can be readily computed with the exception of those for which \( p = q \), in which case \( \theta_p \) is indeterminate. To resolve this indeterminacy, we take advantage of the symmetries of \( G_{ij}^s(x, x') \) and make use of definition (20) to recast (22) as

\[
I = \frac{1}{2} \int_{\Omega'} \int_{\Omega'} \left[ N_{a, x}(x) - N_{a, x}(x') \right] \left[ N_{b, x}(x') - N_{b, x}(x) \right] G_{ij}^s(x, x') \, d\Omega \, d\Omega'
\]

(23)
But since $N_{e,s}^a(x) - N_{e,s}^a(x') \sim O(|x - x'|^3)$ and $N_{b,p}(x) - N_{b,p}(x) \sim O(|x - x'|)$ as $r = |x - x'| \to 0$, it follows that the integrand of (23) vanishes when $x' = x$. Consequently, the contribution of the terms $p = q$ to (22) is zero. The remaining terms in I can be computed without difficulty.

Next we turn our attention to the singular part, II, of (21). We begin by using a conventional quadrature rule to evaluate the integral with respect to, say, $x$, with the result

$$II \approx \sum_p w_p \int_{\Omega'} \frac{\phi_{iajb}(\xi^e_p, \theta)}{|\xi^e_p - x'|} d\Omega'$$  \hspace{1cm} (24)

To evaluate the remaining integral with respect to $x'$ in (24), it proves convenient to introduce a system of polar co-ordinates $(r, \theta)$ centered at $\xi^e_p$. In this co-ordinate system, $|\xi^e_p - x'| = r$, and (24) reduces to

$$II \approx \int_{\Omega'} \frac{\phi_{iajb}(\xi^e_p, \theta)}{|\xi^e_p - x'|} d\Omega' = \int_0^{2\pi} \int_0^{\rho(\theta)} \phi_{iajb}(\xi^e_p, \theta) \frac{r}{r} dr d\theta = \int_0^{2\pi} \phi_{iajb}(\xi^e_p, \theta) r(\theta) d\theta$$  \hspace{1cm} (25)

Finally, for right-sided triangles the integrals with respect to $\theta$ can be computed by recourse to standard formulae.

3.3. Calculation of stress intensity factors

A primary unknown in the analysis of cracks in elastic solids is the distribution of stress intensity factors along the front. The accurate calculation of these quantities is of critical importance for predicting the path of a growing crack. For a three-dimensional crack of arbitrary geometry, the stress intensity factors may be defined with reference to a local Cartesian frame such that the $x_1$-$x_3$ plane is tangent to the crack, with the $x_3$-axis tangent to the crack front, Figure 3. For a linear elastic solid, the opening displacements a small distance $\rho$ behind the tip take the asymptotic form

$$u_i \sim 8\Lambda_{ij}K_j \sqrt{\rho}/2\pi$$  \hspace{1cm} (26)

where

$$[\Lambda_{ij}] = \frac{1}{2\mu} \begin{bmatrix} 1 - \nu & 0 & 0 \\ 0 & 1 - \nu & 0 \\ 0 & 0 & 1 \end{bmatrix}$$  \hspace{1cm} (27)

![Figure 3. Local reference frame for the calculation of stress intensity factors](image)
in the isotropic case. By sampling the opening displacements near the tip of the crack, the stress intensity factors may be extracted directly from (26). However, to obtain accurate values independent of the choice of \( r \), the discretized opening displacements must be of the form (26) sufficiently close to the crack front. This may simply be accomplished by displacing to quarter-point locations the midside nodes of element sides adjacent to the front. The desired parabolic opening profile (26) is thus incorporated into the element interpolation.\(^8\), \(^24\), \(^25\)

3.4. Crack growth

In some applications, the geometry of a propagating crack is not fully known a priori, and has to be determined as part of the analysis. Two classes of problems arise, depending on whether the ‘path’, or surface of growth, of the crack is, or is not, known beforehand. In the former case, the crack is constrained to grow within a predetermined surface, and only the geometry of the front as it propagates through that surface remains to be determined. In some instances, weakly bonded interfaces furnish the likely path of crack growth. In other cases, such as the planar growth of a crack under mode I loading, the path is determined by symmetry.

We begin by considering this class of problems. The geometry of a propagating crack front is depicted in Figure 4. Imagine two infinitesimally close successive configurations of the crack, and let \( da(s) \) denote the distance of advance of the front in the direction normal to itself. Points on the crack front are parametrized by the arc length \( s \). With reference to Figure 4, the ‘kinking angle’ \( \omega(s) \) is defined as the angle made by the normals to the two successive crack fronts. For a smooth crack surface, \( \omega(s) \) is infinitesimal. If, by contrast, the surface of the crack has folds or branching lines, \( \omega(s) \) may attain finite values. In either case, \( \omega(s) \) follows directly from the geometry of the known crack path.

In order to determine \( da(s) \), a fracture criterion needs to be postulated. For brittle solids, a commonly adopted criterion requires that

\[
\mathcal{G}(s) \equiv \frac{1 - v}{2\mu} \left[ K_1^2(s) + K_2^2(s) + \frac{1}{1 - v} K_3^2(s) \right] \leq \mathcal{G}_c
\]

where \( \mathcal{G}(s) \) is the energy release rate at \( s \) and \( \mathcal{G}_c \) is a material constant. When \( \mathcal{G}(s) < \mathcal{G}_c \), the crack front arrests at \( s \). For crack growth to be possible, the condition \( \mathcal{G}(s) = \mathcal{G}_c \) must be satisfied. Situations in which \( \mathcal{G}(s) > \mathcal{G}_c \) are presumed to be physically inadmissible. Upon a change in loading, the crack front geometry may need to be adjusted to enable the foregoing conditions to be satisfied everywhere on the front. In the present setting, a particularly convenient means of

Figure 4. Geometry of the propagating crack front and the local reference frames
enforcing the fracture criterion is to recast it as a pseudodynamic crack-tip equation of motion. Following Freund, the crack-tip velocity of a dynamically running crack is an equation which is closely approximated by the expression

$$\dot{a}(s) = \begin{cases} \frac{c_R}{c_R} \frac{\sigma(s)}{\sigma_c}, & \text{if } \sigma(s) > \sigma_c \\ 0, & \text{if } \sigma(s) < \sigma_c \end{cases}$$

(29)

where $\dot{a} = da/dt$ and $c_R$ is the Rayleigh wave speed. We note that the full range of $\sigma$ is now admissible, and that the crack-tip speed is a function of the extent by which $\sigma(s)$ exceeds $\sigma_c$. For quasistatic crack growth, $\dot{a} \ll c_R$ and the fracture criterion (28) is closely satisfied. Quasistatic crack propagation may be attained by increasing the loads slowly in time, provided that fracture is stable. Alternative forms of the crack-tip equations of motion have been used by Rice and Gao and Rice, Fares, and Bower and Ortiz.

The time-stepping procedure adopted in calculations is modelled after that proposed by Fares. The increment of time is adjusted so that, over one step, the fastest moving point on the front travels a distance of the order of the local mesh size. The remote loads are then scaled so that the maximum value of $\sigma(s)$ is slightly in excess of $\sigma_c$. During stable crack growth, the distribution of $\sigma(s)$ thus computed is nearly uniform throughout the front. Otherwise, points where $\sigma(s)$ is large would accelerate and overtake the rest of the front. This, in turn, would have the effect of lowering $\sigma(s)$ at those points, while increasing it elsewhere, which works to restore the uniformity of $\sigma(s)$. During unstable crack growth, the remote loads need to be steadily reduced for the maximum value of $\sigma(s)$ to remain near $\sigma_c$. This reduction is normally accompanied by extensive crack arrest in those portions of the front where $\sigma(s)$ falls below $\sigma_c$.

In calculations, we remesh after every step to accommodate the changes in geometry. Meshes are constructed by Delaunay triangulation based on the corner nodes of elements, with the midside nodes added subsequently. Curved surfaces are mapped onto a plane before being triangulated. The nodal spacing over the front is continuously adjusted so as to be in direct proportion to the local radius of curvature. Nodes are then distributed over the crack with nodal densities varying inversely with the distance to the front.

A more challenging class of problems concerns situations in which the path of the crack is not known beforehand. Under these circumstances, the fracture criterion must be supplemented with a rule for determining the kinking angle $\omega(s)$, Figure 4. Numerous criteria for determining the kinking angle have been proposed in the literature. For instance, a frequently used criterion is to require that $K_3(s) = 0$ everywhere on the front. This furnishes the requisite condition for determining $\omega(s)$. However, a generally accepted three-dimensional theory of kinking applicable to the case $K_3 \neq 0$ remains to be developed and, therefore, problems of this general class will not be addressed in this paper.

3.5. Periodic and semi-infinite cracks

Applications concerned with micromechanical aspects of fracture often lead to the consideration of cracks of semi-infinite extent. Because of the practical impossibility of meshing out infinite domains by means of ordinary elements, these cracks require special handling. In addition, the microstructure of the solid is frequently idealized as being periodic, which results in semi-infinite cracks exhibiting a periodic variation parallel to the crack front. Here, the aim is to reduce the calculations to one period. In so doing, the interaction-at-a-distance between periods must properly be accounted for.

Consider a crack of periodic geometry in the $x_2$-direction, with period $h$, Figure 5. The tractions acting on the faces of the crack possess the same periodicity. Let $\Omega$ denote the region
occupied by one period of the crack. Then, the potential energy per period of the solid takes the form

\[ \Phi = \lim_{N \to \infty} \sum_{n=-N}^{N} \frac{1}{2} \int_{\Omega} G_{ijkl}(\mathbf{x}, \mathbf{x}', n) u_{i,j}(\mathbf{x}) u_{k,l}(\mathbf{x}') \, d\Omega \, d\Omega' - \int_{\Omega} T_i(\mathbf{x}) u_i(\mathbf{x}) \, d\Omega \]  

(30)

where we have set

\[ G_{ijkl}(\mathbf{x}, \mathbf{x'}'; n) = G_{ijkl}(\mathbf{x}, \mathbf{x'} + nb \mathbf{e}_2) \]  

(31)

For cracks which are bounded in the \( x_1 \)-direction, the convergence of the limiting process implied in (30) may be established as follows. Recall from Section 2 that the opening displacements of the crack may be represented as a continuous distribution of dislocation loops. Because the opening displacements \( \mathbf{u} \) vanish identically at the crack front, it follows that the net Burgers vector contained in one period of the crack must itself vanish identically. Hence, by Saint-Venant's principle, distant periods of the crack see each other as dislocation dipole segments, the interaction between which decays with distance as \( 1/r^2 \) (see Reference 10). Hence, the terms in the sum (30) decay as \( 1/n^2 \) for large \( n \) and the sum converges. The opening displacements then follow by minimization of (30).

Next we show how the analysis of a semi-infinite periodic crack can be reduced to a bounded domain. For simplicity we confine our discussion to cracks which are initially planar and have a straight front. The solid is subjected to remote loads which would induce a standard \( K \)-field were the crack to remain in its initial configuration. In particular the tractions \( T_i \) directly applied on the faces of the crack vanish. The general case can be treated similarly, provided that the opening displacements are known analytically for the initial geometry of the crack. Let the crack initially lie on the \( x_1-x_2 \) plane with its front on the \( x_3 \)-axis, Figure 5. Denote by \( \Omega_0 \) the initial domain of the crack. The crack subsequently grows in the negative \( x_1 \)-direction, possibly out of its plane. We begin by writing the opening displacements as

\[ \mathbf{u} = \mathbf{u}^0 + \hat{\mathbf{u}} \]  

(32)

where \( \mathbf{u}^0 = \mathbf{0} \) for \( x_1 \leq 0 \), and

\[ u_1^0 = \frac{K_{ii}^o}{\mu} \frac{4(1-\nu)}{\sqrt{2\pi}} \sqrt{x_1}, \quad u_2^0 = \frac{K_{ii}^o}{\mu} \frac{4(1-\nu)}{\sqrt{2\pi}} \sqrt{x_1}, \quad u_3^0 = \frac{K_{iii}^o}{\mu} \frac{4}{\sqrt{2\pi}} \sqrt{x_1} \]  

(33)
for $x_1 \geq 0$. In (33), $K_p^x$, $K_p^y$ and $K_p^z$ are the remotely applied stress intensity. The term $u^0$ represents the behaviour of the opening displacements for large $x_1$, i.e. far away from the front. Consequently, the remaining unknown term $\hat{u}$ may be expected to decay quickly to zero for large $x_1$. This situation is exploited by setting $\hat{u} = 0$ for $x_1 \geq L$, i.e. beyond some distance $L$ from the $x_2$-axis. Let $\hat{\Omega}$ denote the support of $\hat{u}$. Within this domain, $\hat{u}$ is interpolated in the manner discussed in Section 3.1. Evidently, the error incurred in restricting $\hat{u}$ to $\hat{\Omega}$ can be made arbitrarily small by taking $L$ sufficiently large. Inserting (32) into (30) and setting $T_i = 0$, the potential energy per period takes the form

$$
\Phi = \lim_{N \to \infty} \sum_{n = -N}^{n = N} \left\{ \frac{1}{2} \int_{\Omega \setminus \hat{\Omega}} G_{ijkl}(x, x', n) \hat{u}_{i,j}(x) \hat{u}_{k,l}(x') \, d\Omega \, d\Omega' + \int_{\Omega \setminus \hat{\Omega}} G_{ijkl}(x, x', n) u^0_{i,j}(x) \hat{u}_{k,l}(x') \, d\Omega \, d\Omega' + \frac{1}{2} \int_{\Omega \setminus \hat{\Omega}} G_{ijkl}(x, x', n) u^0_{i,j}(x) u^0_{k,l}(x') \, d\Omega \, d\Omega' \right\}
$$

(34)

The correction $\hat{u}$ to $u^0$ now follows by minimization of (34). We note that, by requiring $\hat{u}$ to vanish on the boundary of $\hat{\Omega}$, the zero net Burgers vector argument put forth in connection with (30) carries over to the present setting and establishes the boundedness of the first term in (34). The third term is independent of $\hat{u}$ and can, therefore, be disregarded. Finally, we show that the remaining term in (34) can be reduced to an integral over the domain $\Omega - \Omega_0 = \{ x \in \Omega, x_1 < 0 \}$. To this end, let $T_i^\infty$ denote the tractions induced on the surface $\Omega - \Omega_0$ by the opening of the initial half-plane crack under the action of the current remote loads. Clearly, if in addition to the remote loads, one were to apply tractions $- T_i^\infty$ over the region $\Omega - \Omega_0$ of the extended crack, the resulting opening displacements would equal $u^0$. Hence, the first variation about $u^0$ of the potential energy of the solid containing the extended crack subjected to remote loads and tractions $- T_i^\infty$ over $\Omega - \Omega_0$ must vanish identically. This yields the identity

$$
\lim_{N \to \infty} \sum_{n = -N}^{n = N} \left\{ \int_{\Omega \setminus \hat{\Omega}} G_{ijkl}(x, x', n) u^0_{i,j}(x) \delta u_{k,l}(x') \, d\Omega \, d\Omega' \right\} + \int_{\Omega - \Omega_0} T_i^\infty(x') \delta u_i(x') \, d\Omega' = 0
$$

(35)

for any virtual opening displacements $\delta u_i$ satisfying the essential boundary condition $\delta u_i = 0$ on the crack front. Setting $\delta u_i = \hat{u}_i$ in (35) and recalling that $\hat{u}_i$ is supported on $\hat{\Omega}$, one finds

$$
\lim_{N \to \infty} \sum_{n = -N}^{n = N} \left\{ \int_{\Omega \setminus \hat{\Omega}} G_{ijkl}(x, x', n) u^0_{i,j}(x) \hat{u}_{k,l}(x') \, d\Omega \, d\Omega' \right\} + \int_{\Omega - \Omega_0} T_i^\infty(x') \hat{u}_i(x') \, d\Omega' = 0
$$

(36)

Substitution of this identity into (34) finally yields

$$
\Phi = \lim_{N \to \infty} \sum_{n = -N}^{n = N} \left\{ \frac{1}{2} \int_{\Omega \setminus \hat{\Omega}} G_{ijkl}(x, x', n) \hat{u}_{i,j}(x) \hat{u}_{k,l}(x') \, d\Omega \, d\Omega' - \int_{\Omega - \Omega_0} T_i^\infty(x') \hat{u}_i(x') \, d\Omega' \right\}
$$

(37)

where we have omitted inconsequential constants. It bears emphasis that all integrals in (37) have bounded domains contained in the region covered by the finite element mesh. It is also interesting to note that, if the crack remains in its initial configuration, then $\Omega - \Omega_0$ is empty and the forcing term drops out of (37). This in turn gives $\hat{u} = 0$ and, from (32), $u = u^0$, as expected.

A final practical consideration concerns the computation of the second integral in (37). Because of the square-root singularity in $T_i^\infty$ and the limited spatial resolution afforded by the mesh, a direct computation of the effective nodal forces (16) tends to produce irregular opening displacements in the neighbourhood of the $x_1$-axis. The quality of the solution can be improved substantially by computing the tractions $T_i^\infty$ from the stress field of a Dugdale–Barenblatt crack.
By a suitable choice of the cohesive zone width, solutions are obtained which are considerably smoother than those resulting from a standard $K$-stress field. General analytical expressions for the stress field of a Dugdale–Barenblatt crack are given, e.g. by Ortiz and Blume.\textsuperscript{29} In the special case of a crack growing within its plane in pure mode I one has

$$
T^\pi_{33}(x_1) = \begin{cases} 
\frac{K^\infty_1}{\sqrt{2\pi R}} \arcsin \left( \frac{-R}{x_1} \right), & x_1 \leq -R, \\
\frac{\pi}{8R} K^\infty_1, & -R \leq x_1 \leq 0, \\
0, & x_1 > 0 
\end{cases}
$$

(38)

for the only non-zero component of $T^\pi$, and

$$
u_0^0(x_1) = \begin{cases} 
\frac{K^\infty_1}{\sqrt{2\pi R}} \frac{1-v}{\mu} \left[ \sqrt{R(R+x_1)} + x_1 \ln(\sqrt{R} + \sqrt{R + x_1}) - \frac{x_1}{2} \ln|x_1| \right], & -R \leq x_1 \\
0, & x_1 \leq -R 
\end{cases}
$$

(39)

for the only non-zero component of $u^0$. Here $R$ denotes the width of the cohesive zone which, in the present context, is an adjustable parameter. In calculations, we have taken $R$ to equal twice the element size.

4. TEST SOLUTIONS

In this section we collect several test examples which demonstrate the accuracy and versatility of the method. The examples concern: penny-shaped and elliptical cracks under uniform remote stresses; a spherical cap crack under uniform uniaxial and hydrostatic tension; a circular arc crack subjected to uniform remote stresses; a semi-infinite crack with sinusoidal crack front opening under the action of mode I remote loading; and a semi-infinite crack growing in mode I through a square array of circular obstacles. We take the known analytical solutions for the penny-shaped and elliptical cracks as a basis for assessing the convergence properties of the method. The spherical cap and circular arc cases test the accuracy of the method in applications involving non-planar geometries. In particular, the known analytical solution for a circular arc crack permits a detailed accuracy assessment. The example of a half-plane crack with a sinusoidal front illustrates how the analysis of semi-infinite periodic cracks can be restricted to bounded domains. Finally, the analysis of crack trapping and bridging by an array of obstacles demonstrates the front marching and adaptive meshing aspects of the method.

4.1. Penny-shaped and elliptical cracks

Consider a penny-shaped crack of radius $a$ in an unbounded elastic solid acted upon by uniform remote normal and shear tractions, $\sigma^\infty_{33}$ and $\sigma^\infty_{32}$, respectively, Figure 6. The opening displacements are known to be of the form\textsuperscript{30}

$$
u_1 = 0$$

$$
u_2 = \frac{16(1-v^2)}{\pi E(2-v)} \sqrt{a^2 - r^2} \sigma_{33}^\infty$$

$$
u_3 = \frac{8(1-v^2)}{\pi E} \sqrt{a^2 - r^2} \sigma_{33}^\infty$$

(40)
while the stress intensity factors along the crack front are

\[ K_1 = \frac{2}{\pi} \sqrt{\pi a \sigma_{33}^\infty} \]

\[ K_{II} = \frac{4}{\pi(2 - v)} \sin \theta \sqrt{\pi a \sigma_{23}^\infty} \]

\[ K_{III} = \frac{4(1 - v)}{\pi(2 - v)} \cos \theta \sqrt{\pi a \sigma_{33}^\infty} \]  \hspace{1cm} (41)

We take \( v = 0.3 \) in all subsequent calculations.

The mesh used in the analysis is depicted in Figure 7. The circumferential mesh lines divide the range of the function \( \sqrt{a^2 - r^2} \) into equal intervals. Then, the innermost circumferential mesh line is divided into eight segments, the next circumferential mesh line into 16 segments, and so on. This defines the positions of the corner nodes of the elements. The mesh is then constructed by Delaunay triangulation, and the mid-side nodes are added subsequently. Meshes of varying numbers of elements are constructed in this manner.

Typical computed opening profiles are shown in Figure 8 against the exact solution. As a measure of the error, for each displacement component we define

\[ E_i = \frac{|u_{im} - u_i|}{u_i} \]  \hspace{1cm} (42)

where \( u_{im} \) and \( u_i \) are the numerical and exact opening displacements at the centre of the crack, respectively. The variation of \( E_i \) with the number of elements \( N_e \) in the mesh is shown in Figure 9. Errors in the tangential displacement are consistently smaller than errors in the normal displacement. Clearly, the average mesh size \( h \) varies as \( 1/\sqrt{N_e} \). Relative to the simple error measure adopted, the rate of convergence of the solution can be defined as the asymptotic value of the
logarithmic derivative $\alpha_i = d(\log E_i)/d(\log h)$ as $N_e \to \infty$. Then, it follows that $E_i \sim h^\alpha$ asymptotically. The numerical data plotted in Figure 9 suggest that the rate of convergence of the method is approximately quadratic, i.e. $\alpha_i \approx 2$.

Stress intensity factors are computed from the opening displacements in the manner outlined in Section 3.3. The opening displacements are sampled at the interior corner node of elements having one side on the front. The maximum error in the stress intensity factors computed from the opening displacements displayed in Figure 8, which correspond to a 752 element mesh with 484 elements on the front, is less than 1 per cent. As noted by Cruse, the use of transition elements, with midside nodes displaced to quarter-point locations, greatly improves the accuracy of the computed stress intensity factors.
Next we consider an elliptical crack of semiaxes $a$ and $b$ in the ratio $k' = b/a$. The crack opens under the action of uniform remote normal and shear stresses, $\sigma_{33}^x$ and $\sigma_{23}^x$, respectively. The latter stress is applied in the direction of the minor axis. Points on the crack front may be parametrized by the polar angle $\theta$ as $x_1 = a \cos \theta$ and $x_2 = b \sin \theta$. The stress intensity factors are then given by\(^{31}\)

$$K_I = \frac{\sigma_{33}^x}{E(k)} \sqrt{\frac{\pi b}{E(k)}} \left( \sin^2 \theta + k'^2 \cos^2 \theta \right)^{1/4}$$

$$K_{II} = \frac{\sigma_{23}^x}{(k'^2 + v k^2) E(k)} \frac{\sin \theta}{k'^2 K(k) \sqrt{\sin^2 \theta + k'^2 \cos^2 \theta}}$$

$$K_{III} = \frac{\sigma_{23}^x}{(k'^2 + v k^2) E(k)} \frac{\cos \theta}{(1 - v) k'^2 K(k) \sqrt{\sin^2 \theta + k'^2 \cos^2 \theta}}$$

(43)

where $k^2 = 1 - k'^2$ and

$$K(k) = \int_{0}^{\pi/2} \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}}, \quad E(k) = \int_{0}^{\pi/2} \frac{1}{\sqrt{1 - k^2 \sin^2 \phi}} d\phi$$

(44)

are the elliptic integrals of the first and second kinds, respectively.

Figures 10–12 compare the analytical and computed stress intensity factors for cracks of aspect ratios $a/b = 1, 2$ and 3. The mesh used in the computations comprises a total of 752 elements, with 484 elements on the front, in the case $a/b = 1$, with numbers varying slightly in the other two cases. Stress intensity factors are normalized by $K_0 = (2/\pi) \sqrt{\pi b \sigma_{33}^x}$. In all cases the largest numerical error, which is incurred in the computation of $K_1$, is less than 1 per cent.

4.2. Circular-arc and spherical cracks

As a first test of the ability of the method to deal with three-dimensional geometries, we consider the case of a cylindrical crack having a circular-arc cross-section, Figure 13. The crack is
subjected to uniform remote stresses contained in its cross-section, which renders the problem two dimensional. The stress intensity factors are given by Cotterell and Rice\textsuperscript{15} as

\[
K_1 = \sqrt{\pi a} \left\{ \left( \sigma_{22}^{\infty} + \sigma_{11}^{\infty} \right) - \left( \frac{\sigma_{22}^{\infty} - \sigma_{11}^{\infty}}{2} \right) \sin^2 (\alpha/2) \cos^2 (\alpha/2) \right\} \frac{\cos(\alpha/2)}{[1 + \sin^2 (\alpha/2)]} + \left( \frac{\sigma_{22}^{\infty} - \sigma_{11}^{\infty}}{2} \right) \cos (3\alpha/2) - \sigma_{12}^{\infty} [\sin (3\alpha/2) + \sin^3 (\alpha/2)] \right \}
\]

(45)
Figure 12. Variation of $K_{II}$ for elliptical crack

Figure 13. Circular-arc crack

$$K_{II} = \sqrt{\pi a} \left\{ \left[ \left( \frac{\sigma_{22}^{+} + \sigma_{11}^{+}}{2} \right)^{2} - \left( \frac{\sigma_{22}^{-} - \sigma_{11}^{-}}{2} \right)^{2} \right] \sin^{2} \left( \frac{\alpha}{2} \right) \cos^{2} \left( \frac{\alpha}{2} \right) \right\} \frac{\sin \left( \frac{\alpha}{2} \right)}{\left[ 1 + \sin^{2} \left( \frac{\alpha}{2} \right) \right]}$$

$$+ \left( \frac{\sigma_{22}^{-} - \sigma_{11}^{-}}{2} \right) \sin \left( 3\alpha/2 \right) + \sigma_{12}^{+} \left[ \cos \left( \alpha/2 \right) + \cos \left( \alpha/2 \sin^{2} \left( \alpha/2 \right) \right) \right] \right\}$$

(46)

where $2a$ is the distance between the tips of the crack and $2\alpha$ is the angle subtended by the arc, Figure 13.

Despite the two-dimensional character of the problem, in our calculations we discretize a finite strip of the crack, which is treated as being periodic. The mesh employed in the calculations
contains a total of 304 elements, with 60 elements on each front, Figure 14. The series in (37) is summed up to 200 terms. Calculations were carried out for remote stresses $\sigma_{11}^\infty = \sigma_{22}^\infty = 0$. Stress intensity factors are normalized by $K_0 = \sigma_{22}^\infty \sqrt{\pi a}$. Figure 15 shows a comparison between computed and exact stress intensity factors for $\alpha$ in the range $[0^\circ, 75^\circ]$. Beyond the angle $\alpha \approx 76.5^\circ$, closure sets in at the tips. The good accuracy of the method is evident from Figure 15.

A second test involving a surface of double curvature is furnished by a spherical crack bounded by a circular front subjected to uniform remote stresses, Figure 16. A physical situation leading to the formation of such cracks consists of the decohesion of an equiaxed second-phase particle. Meshes are constructed by mapping the penny-shaped-crack mesh described in Section 4.1 onto a sphere, Figure 17. The computed variation of the stress intensity factors with subtended angle $\alpha$ is shown in Figure 18 for the cases of hydrostatic tension and uniaxial tension along the axis of symmetry. As expected, in both cases $K_1$ decreases steadily with $\alpha$. In the case of uniaxial tension, $K_1$ falls off to zero at $\alpha \approx 75^\circ$, beyond which point crack-tip closure sets in. Although exact solutions for spherical cracks are not available at present, the numerical solution of Sladek and Sladek offers a basis for comparison. The good agreement between Sladek and Sladek's solution and the present calculations is evident from Figure 18.

![Image of mesh used in the analysis of circular-arc crack](image)

**Figure 14.** Example of the mesh used in the analysis of circular-arc crack

![Graph of stress intensity factors](image)

**Figure 15.** Stress intensity factors for a circular-arc crack
Figure 16. Spherical crack

Figure 17. Example of mesh used in the analysis of spherical cracks

Figure 18. Stress intensity factors for a spherical crack
4.3. Sinusoidal half-plane crack

As a test of the procedure for analysing periodic semi-infinite cracks given in Section 3.5, we consider the case of a semi-infinite crack with a sinusoidal front opening in mode I. The crack is infinite in the $x_1$-direction and periodic in the $x_2$-direction. The period of the front waviness is $b$ and its amplitude $A$. The finite element mesh used in the analysis, which contains a total of 443 elements with 96 elements on the front, is shown in Figure 19. The mesh extends in the $x_1$-direction over a distance equal to 500$b$. The series in (37) is truncated after 150 terms. For a straight-sided crack, the applied stress intensity factor $K^\infty_1$ is recovered to within 0.5 per cent, which provides an indication of the overall accuracy of the calculations.

The problem of a sinusoidal half-plane crack has been solved numerically by Fares,\textsuperscript{18} and by Bower and Ortiz.\textsuperscript{19} The present calculations are in good agreement with their results. Although the exact solution for a sinusoidal half-plane crack is not known, Bower and Ortiz\textsuperscript{19} have pointed out that the identity

$$\langle (K_1/K^\infty_1)^2 \rangle = \left[ \frac{1}{b} \int_{-b/2}^{b/2} \left( \frac{K_1(x_2)}{K^\infty_1} \right)^2 \, dx_2 \right]^{1/2} = 1 \quad (47)$$

is a necessary consequence of the path invariance of the $J$-integral. This provides a means of assessing the accuracy of the solution. The computed value of $E \equiv \langle (K_1/K^\infty_1)^2 \rangle - 1$ is shown in Figure 20 for several amplitudes of the front waviness. Errors tend to increase with amplitude $A$, but this effect is partly attributable to mesh distorsion and could be mitigated by mesh optimization.

4.4. Trapping and bridging by a square array of tough particles

The preceding examples have been concerned with questions of accuracy in the computation of opening displacements and stress intensity factors for a fixed crack geometry. Our final example tests the crack extension procedure discussed in Section 3.4. We consider an initially straight half-plane crack propagating through a solid of toughness $K^\text{mat}_{IC}$ reinforced by a rectangular array of impenetrable circular particles. The elastic properties of the particles are assumed to be identical to those of the matrix. The solid is subjected to remote loads acting so as to induce

![Figure 19. Half-plane crack with sinusoidal front](image-url)
a mode I stress intensity factor on the crack front. The magnitude of the applied load is parametrized by a remote stress intensity factor $K_0$, as discussed in Section 3.5.

The mechanism by which the crack propagates through the array of obstacles is presently well understood\textsuperscript{17–20} and is shown schematically in Figure 21. The front successively bows out between the particles, coalesces with itself in their wake, and advances through the intervening matrix to become trapped by the next row of obstacles, where the same sequence of events repeats itself. Every bypassed row of obstacles is added to the wake of the crack as a row of pinning particles. The simple case of impenetrable obstacles considered here represents the limiting behaviour of a composite reinforced by particles or fibres of infinite toughness. The case of particles of finite toughness has been analysed in detail elsewhere.\textsuperscript{17,20}

Figure 22 depicts the geometry of the problem. Because of the regularity of the arrangement of obstacles, the crack is periodic in the $x_2$-direction with period $b$. As shown in Section 3.5, the domain of analysis can then be restricted to a bounded domain. The ratio of particle radius to

![Figure 20. Stress intensity factors for a sinusoidal half-plane crack](image1)

![Figure 21. Crack bypassing an obstacle](image2)
spacing is \( R/b = 0.25 \). The crack front is initially straight and just short of the first array of particles. As the remote stress intensity factor \( K_1^e \) is raised ever so slightly over the toughness of the matrix, the crack proceeds forward and comes in full contact with the obstacles. The propagation of the front is governed by the crack-tip equation of motion (29). The parameter \( c_R \) in (29) representing the Rayleigh wave speed has no effect on the calculations other than to set the time scale. The analysis proceeds incrementally by the time-stepping procedure discussed in Section 3.4. Nodes coming in contact with the obstacles arrest and are subsequently held fixed.

Figure 23 shows the mesh after the crack has bypassed one row of obstacles and is about to bypass another. The successive crack front profiles are plotted in Figure 24 together with the corresponding values of the applied stress intensity factor. The variation of applied stress intensity factor with the average distance advanced by the crack (i.e. the area swept by the front per unit distance in the \( x_2 \)-direction) is shown in Figure 25. As is evident from these figures, \( K_1^e \) must be steadily increased for the crack to bow through the first row of obstacles. At a critical configuration, however, the attraction between the two lobes of the crack approaching each other in the wake of the obstacles overcomes the fracture resistance of the matrix and the crack begins to grow unstably under decreasing applied load. The maximum value of \( K_1^e \) at the critical configuration is computed to be 2.36 \( K_{IC}^{mat} \). Figures 24 and 25, which is nearly identical to the value of 2.35 \( K_{IC}^{mat} \) given by Fares, and slightly lower than the value of 2.49 \( K_{IC}^{mat} \) computed by Bower and Ortiz. The elevation in applied stress intensity factor over the toughness of the matrix required to bypass the first row of particles may be thought of as a measure of the toughening effect of the 'trapping' mechanism. Throughout the stable regime, the stress intensity factors \( K_1(s) \) remains close to \( K_{IC}^{mat} \) at all points \( s \) of the front contained in the matrix. By contrast, the value of \( K_f(s) \) at points in contact with the particles rises sharply over \( K_{IC}^{mat} \). \( K_f(s) \) attains its
maximum at the point $s$ of first contact between the front and the particles, and rises to $K_{I, \text{max}} = 3.46 \, K_{IC}^{\text{mat}}$ at peak load. This value compares well with that of $3.52 \, K_{IC}$ given by Fares,\textsuperscript{18} while being somewhat lower than the value of $3.73 \, K_{IC}^{\text{mat}}$ determined by Bower and Ortiz.\textsuperscript{19}

A similar sequence of events takes place after the crack makes contact with the second array of obstacles, Figures 23 and 24. Initially, the crack grows stably under increasing load up to a maximum of $K^\infty_I \approx 3.47 \, K_{IC}^{\text{mat}}$, which is in good agreement with the value of $3.43 \, K_{IC}^{\text{mat}}$ computed by Bower and Ortiz.\textsuperscript{19} The difference between the two successive maxima of $K^\infty_I$ reflects the stabilizing effect of the first row of particles bypassed by the crack, and may be construed as a measure of the toughening effect of the 'bridging' mechanism. As before, the points of the front in contact with the second row of particles experience a sharp increase in the local value of the stress intensity factor, which attains a maximum of $5.49 \, K_{IC}^{\text{mat}}$ at peak load.

Between these two stable regimes, the crack grows unstably under decreasing remote loads. Extensive crack arrest takes place during the unstable regime. Bower and Ortiz\textsuperscript{20} have noted that, if the fracture criterion (28) is strictly enforced, then $K^\infty_I$ must necessarily decrease to zero at the point of coalescence. Because of the use of a crack-tip equation of motion in place of a fracture criterion, this limit is not exactly attained in our calculations. The computed value of $K^\infty_I$ does drop precipitously, however, during the unstable regime, Figures 24 and 25, and reaches a minimum at the point of coalescence. The drop in $K^\infty_I$ is rendered increasingly sharper the closer $K_I(s)$ is forced to remain with $K_{IC}^{\text{mat}}$ in the propagating portions of the front. The ability of the method to determine the crack front configuration during the unstable regime is noteworthy.
5. SUMMARY AND CONCLUSIONS

A finite element methodology for analysing propagating cracks of arbitrary three-dimensional geometry has been developed. By representing the opening displacements of the crack as a distribution of dislocation loops and minimizing the potential energy of the solid, governing
integral equations are obtained which are only mildly singular. A simple quadrature scheme then suffices to compute all the element arrays accurately. Our formulation generalizes that of Bui and Clifton and Abou-Sayed, who restricted their attention to planar cracks under symmetric mode I loading. Because of the variational basis of the method adopted here, and in Reference 3, the resulting system of equations is symmetric. The method also shares this desirable property with the Galerkin boundary element formulation of Maier and Polizzotto.

By employing six-noded triangular elements and displacing midside nodes to quarter-point positions, the opening profile near the front is endowed with the correct asymptotic behaviour. This enables the direct computation of stress intensity factors from the opening displacements. We have addressed in some detail the special but important cases of periodic and semi-infinite cracks. We show that the analysis of periodic cracks can be restricted to one period, and that semi-infinite cracks need only be discretized over a finite region. Finally, the geometry of propagating cracks is updated incrementally according to a pseudodynamic crack-tip equation of motion. The mesh is continuously adapted to accommodate the changes in geometry of the crack.

The performance of the method has been assessed by means of selected numerical examples. Satisfactory accuracy in opening displacements and stress intensity factors is obtained with practical meshes. A convergence study points to a quadratic rate of convergence of the method. As noted by Cruse, the use of transition elements near the front is seen to greatly enhance the accuracy of the computed stress intensity factors. The crack growth procedure has been tested for the case of a half-plane crack propagating in mode I through a square array of impenetrable particles. The computed crack front profiles and the variation of the applied stress intensity factor with crack advance are in good agreement with the earlier calculations of Fares and Bower and Ortiz. The ability of the method to march the crack through the unstable regime as the crack coalesces with itself in the wake of the particles is noteworthy.

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APPENDIX: SOME ELEMENTS OF SURFACE GEOMETRY
Let $S$ be a surface parametrized by curvilinear co-ordinates $\theta^\alpha$, $\alpha = 1, 2$. Let $a_\alpha$, $a^\gamma$ and $a_{\alpha\beta} = a_\alpha \cdot a_\beta$ denote the corresponding contravariant, covariant bases and the metric tensor, respectively (see, e.g. References 32, 33). Identify $a_3 = a^1$ with the unit normal vector $S$. An opening displacement field $u$ over $S$ admits the representations

$$u = U_i e_i = u_i a_i = u^i a_i \tag{48}$$

where $e_i$ is an orthonormal Euclidean basis, $U_i$ is the corresponding components of $u$, and $u^i$ and $u_i$ are the contravariant and covariant components of $u$ relative to the surface co-ordinate system, respectively. The rate of change of $u$ with respect to $\theta^\alpha$ defines another vector field $u_{,\alpha} = u^i_{,\alpha} a_i$, over $S$ of components

$$u^\beta_{,\alpha} = u^\beta_{,\alpha} + u^\beta \Gamma^\gamma_{\alpha \alpha} - u^3 b^\beta_{,\alpha} \tag{49}$$

where $\Gamma^\gamma_{\alpha \beta}$ are the Christoffel symbols, $b_{\alpha\beta}$ is the curvature tensor (see, e.g. References 32 and 33), and $(\cdot)_{,\alpha}$ denotes partial differentiation with respect to $\theta^\alpha$. The differential operator defined by (49) is the covariant derivative of $u$. Writing $\nabla U_i = U_{i,\alpha} a^\alpha$, lengthy but straightforward computations
yield the identities

\[
\begin{align*}
\left[ \mathbf{e}_j \cdot (\mathbf{n} \times \nabla U_j) \right]_2 &= \left[ \left( \mathbf{a}_j \right)_2 \cdot (\mathbf{u}^j_{\varepsilon} \varepsilon^{\alpha\beta} \mathbf{a}_\beta) \right]_2 \\
\left[ \mathbf{e}_j \cdot (\mathbf{n} \times \nabla U_j) \right]_2 &= (\mathbf{u}^j_{\varepsilon})_2 \left( \mathbf{a}_j \right)_2 \\
\mathbf{e}_j \times \mathbf{a}_3 &= \mathbf{a}_3 \wedge \mathbf{a}_x
\end{align*}
\]

Equation (11) then follows readily from (10) by recourse to identities (50–52).

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