Adaptive mesh refinement in strain localization problems

M. Ortiz and J.J. Quigley, IV
Division of Engineering, Brown University, Providence, RI, USA

Received 30 August 1990
Revised manuscript received 20 February 1991

An adaptive meshing method tailored to problems of strain localization is given. The adaption strategy consists of equi-distributing the variation of the velocity field over the elements of the mesh. A heuristic justification for the use of variations as indicators is advanced, and possible connections with interpolation error bounds are discussed. Meshes are constructed by Delaunay triangulation. It is shown how the Hu–Washizu principle determines a consistent transfer operator for the state variables. Examples of application are given which demonstrate the versatility of the method.

1. Introduction

Problems of strain localization often involve two or more well-differentiated length scales: a global scale commensurate with the macroscopic features of the solid; and a local or microscopic scale of the order of the shear band thickness. In many cases of interest, the micro-scale steadily collapses in time. Depending on the level of modeling, this process of collapse can be self-limiting or it can continue ad infinitum. Examples of the former type of behavior are found in solids with an intrinsic characteristic length, such as non-local and polar continua, thermally conducting plastic solids with thermal softening, solids with cohesive interfaces, and others. By way of contrast, rate-dependent solids lacking an internal length exhibit shear bands for which the collapse of the microscale may not be self-limiting. In either case, it may be of interest to resolve the solution at both the macro and microscales concurrently. Because the time-dependent location, shape and structure of the shear band are generally not known beforehand, methods of solution incorporating some form of mesh adaptivity naturally suggest themselves.

Whereas adaptive methods for linear elasticity and certain classes of nonlinear problems have been developed to an advanced state (see, e.g., [1]), there is comparatively much less experience on the design and application of mesh adaption techniques to problems involving history dependent solids. The paucity of studies is even more acute in the area of strain localization (for a notable exception, see [2]). Several difficulties arise in dealing with these problems which are not a concern in other areas of application. Firstly, the history-dependent character of the constitutive response necessarily renders the problem incremental in time. As the mesh is adapted, the solution cannot be re-computed from scratch, as in the case of the
elastic solid, but has to be continued from a prior state. In particular, some suitable means for transferring the state variables between meshes, or ‘transfer operator’, needs to be devised.

A deeper source of difficulty peculiar to problems of strain localization is the fact that, at the point of inception of localization, the governing equations lose, or nearly lose, ellipticity. Consequently, methods of a posteriori error estimation based on elliptic estimates break down. In essence, the reason for the breakdown is that local residuals pollute the incremental solution out to infinity along the emerging characteristics. By contrast, methods based on the minimization of interpolation error do not rely on the ellipticity of the equations and, therefore, are better suited to localization problems. As is well known \([3, 4]\), the optimality condition for interpolation error minimization is that some suitable norm of the solution be equi-distributed over the elements in the mesh. The appropriate choice of norm depends critically on the nature of the space of solutions for the problem under consideration.

Here, we argue heuristically that localized solutions may reasonably be expected to be of bounded variation, and, hence, a suitable adaption indicator is the variation of the solution over the element. The method may be thought of as being ‘resolution driven’ in that elements are targeted for refinement when the variation of the solution within them is determined to be too high for the interpolation to adequately resolve it. All meshes considered here are composed of triangular elements and are constructed by Delaunay triangulation. When an element is identified as requiring refinement, nodes are added to the mesh at the midpoints of the element sides, and the connectivity is redefined for the entire mesh. We use a Delaunay triangulation algorithm devised by Sloan \([5]\) which asymptotically requires an average of only \(O(N^{5/4})\) operations, \(N\) being the number of nodes in the mesh. We also show how the Hu–Washizu principle can be invoked to determine a canonical transfer operator for the state variables. For the transfer operator to be well-behaved it is critical that the state variable interpolation be as ‘noise-free’ as possible. The six-noded element with linear state interpolation based on three quadrature points per element has been found to be particularly valuable in this respect. Numerical examples are given in Section 6 which illustrate the versatility of the method.

2. Strain localization in solids

We consider time-dependent deformations which carry points \(X\) on some reference configuration \(B_0\) to positions \(x = \phi(X, t)\) on the current configuration \(B_t\). Let \(F\) denote the deformation gradient tensor and let stresses be measured by the first Piola–Kirchhoff stress tensor \(P\). The equations of equilibrium and compatibility take the form

\[
P_{ij,j} + \rho_0 (B_t - A_i) = 0 \quad \text{in} \ B_0, \quad P_{ij} N_j = \bar{T}_i \quad \text{on} \ \Gamma_{r0},
\]

\[
F_{ij} = \phi_{ij} \quad \text{in} \ B_0, \quad \phi_i = \bar{\phi}_i \quad \text{on} \ \Gamma_{u0},
\]

where \(\rho_0\) is the mass density per unit undeformed volume, \(B\) are the body forces, \(A = \dot{\phi}\) the Lagrangian accelerations, \(\bar{T}\) are the tractions applied on the traction boundary \(\Gamma_{r0}\), \(\bar{\phi}\) are the displacements prescribed on the displacement boundary \(\Gamma_{u0}\), and \(N\) is the unit normal on \(\Gamma_{r0}\). We shall adopt upper and lower case indices to designate components with respect to reference frames, respectively.

Start by computing constitutive

\[
P_i = \bar{P}_i.
\]

For a given value of \(\bar{F}\) the loading and incremental

\[
(L)
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where we have assumed (3) is complete.

Next we consider disturbances corresponding consideration

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where \(i = \sqrt{c^2 - 1}\), solutions, \(i\) solutions ste that \(c^2 > 0\), waves, there for stationary

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The local incremental remarked by the loss of elliptic the emerging are not admit pair up to the corresponde lar block \([9]\).

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reference frames associated with the reference (material) and the current (spatial) configurations, respectively.

Start by considering rate-independent solids characterized by piece-wise linear incremental constitutive relations of the form

\[ \dot{P}_{ij} = L_{i,jk,l} \dot{F}_{k,l} . \]  

(2)

For a given state, the incremental moduli \( L \) may exhibit several branches, depending on the value of \( \dot{F} \). For example, in classical theories of plasticity \( L \) has two branches: one for plastic loading and another for elastic unloading. Inserting (2) into the rate form of (1), the rate or incremental problem can be expressed as

\[ (L_{i,jk,l} V_{k,l},_j) = \rho_0 \dot{V}_i , \]

(3)

where we have written \( V = \dot{\phi} \) for the Lagrangian velocity field, and have set \( B = 0 \). Equation (3) is complemented with the appropriate rate form of the boundary conditions in (1).

Next we consider an infinite homogeneous solid undergoing continuing homogeneous deformations, and seek to characterize the response of the solid to small superimposed disturbances. This problem can be investigated for an incrementally linear solid having moduli corresponding to the active branch for continued homogeneous deformation, i.e., the linear comparison solid of Hill [6]. The small wave disturbances are written in the form

\[ V_i = C_i \exp[i(\mathbf{K} \cdot \mathbf{X} - \kappa \omega t)] , \]

(4)

where \( i = \sqrt{-1} \) and \( \kappa = |\mathbf{K}| \). Following Hill [7], we investigate the existence of stationary wave solutions, i.e., solutions of the type (4) with \( \omega = 0 \). The significance of stationary wave solutions stems from their signalling the onset of instability. When all wave speeds are such that \( c^2 > 0 \), then there is stability with respect to small perturbations. When \( c^2 < 0 \) for some waves, there is exponential growth of small disturbances, as noted by Rice [8]. The condition for stationary waves to exist is that

\[ \det(L_{i,jk,l} N_j N_k) = 0 \]

(5)

for some unit vector \( N \).

The localization condition (5) admits several alternative interpretations. Recall that the incremental problem (3) is elliptic if \( \det(L_{i,jk,l} N_j N_k) > 0 \) for all unit vectors \( N \). Therefore, as remarked by Hill [7], the onset of localization in the rate-independent solid coincides with the loss of ellipticity of the incremental problem. The unit vectors \( N \) for which (5) is met define the emerging characteristic directions. Such directions can carry strain discontinuities which are not admissible when the equations are elliptic. Then, two lines of strain discontinuity can pair up to form a shear band. The shear band solutions characterized by (5) are in correspondence with the short wave-length limit of bifurcation solutions for a finite rectangular block [9].

Because the governing equations lose ellipticity at localization, the incremental boundary value problem becomes ill-posed. In finite solids, however, ill-posedness may also occur while
the equations remain elliptic, as a consequence of the failure of the complementing condition [10]. Much as in Hill's localization analysis, the failure of the complementing condition is equivalent to the existence of stationary Rayleigh waves at the boundary [10]. Needleman and Ortiz [11] have emphasized the fact that the failure of the complementing condition furnishes an upper bound for the point of inception of shear banding at free boundaries. An analysis of the complementing condition also determines the geometry of shear bands intersecting a free boundary. The analysis has been extended to perfectly bonded interfaces [11], where the appropriate critical condition is the emergence of stationary Stoneley waves, and to decohering interfaces [12]. In this latter case, the cohesive law for the interface introduces a physical length in the problem which sets the length scale for the solution.

In contrast to rate-insensitive solids, the equations governing the deformation of rate-dependent solids always remain elliptic. In this case, localization instabilities manifest themselves as exponentially growing unstable modes of shear deformation. Because ellipticity is never lost, geometrical defects introduce a length scale into the problem [13]. This is not to say, however, that the ill-posedness problem is entirely eliminated. If, as is common in numerical treatments of plasticity, time is discretized by recourse to a state-update algorithm, the resulting incremental equations do lose ellipticity under the appropriate conditions. The same is true of the equations obtained by a Laplace transform in time of (3). In this case, the critical conditions for the growth of perturbations depend on the rate of growth, with the infinitely slow, or quasistatic, solutions occurring earliest [14].

3. Adaption criteria for localization problems

Because the governing equations lose, or nearly lose, ellipticity at localization, methods of local error estimation based on elliptic estimates break down. Insight into this problem may be gained by means of the following example. Consider a solid deforming in antiplane shear relative to the $x_1-x_2$ plane. Let the solid obey $J_2$-deformation theory of plasticity with power-law hardening. The equation governing the out-of-plane displacement $w(x_1, x_2)$ is

$$\nabla \cdot (|\nabla w|^{(1-n)/n} \nabla w) = f(x_1, x_2),$$

where $n$ is the hardening exponent and $f$ the body force. Next, consider the incremental problem about a uniform state of deformation $w^0_\alpha = \gamma^0 \delta_{\alpha \alpha}$. Linearization of (6) gives

$$\left(\frac{1}{n}\right) u_{,11} + u_{,22} = r(x_1, x_2),$$

where $u$ denotes the incremental displacement field and $r$ the residual forces. Evidently, (7) loses ellipticity when $n \to \infty$, i.e., in the ideally plastic limit. Of particular interest here is the behavior of solutions as the elliptic boundary is approached. Let $G(x_1, x_2)$ be the fundamental solution obtained by setting $r = -\delta(x_1, x_2)$ in (7). It is readily verified that

$$G = -(1/4\pi) \log(nx_1^2 + x_2^2).$$

The corresponding effective shear strain is computed to be

$$|\nabla G| = (1/2\pi) \sqrt{n^2 x_1^2 + x_2^2 / (nx_1^2 + x_2^2)}.$$
This solution is depicted in Fig. 1. It is evident from the figure that, as $n \to \infty$, the level contours of $|\nabla G|$ spread out to infinity along the characteristic direction $x_2$. Consequently, the effect of local perturbations is felt at increasingly large distances from the point of occurrence of the disturbance.

Now, imagine a residual field $r$ with support on a small domain $\mathcal{D}$. The incremental methods of lem may be plane shear sticity with $x_2$ is

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\text{(6)}
\end{equation}

incremental gives

\begin{equation}
\text{(7)}
\end{equation}

identically, (7) here is the fundamental

\begin{equation}
\text{(8)}
\end{equation}

\begin{equation}
\text{(9)}
\end{equation}

Fig. 1. Contour plots of effective shear strain for increasing values of the hardening exponent $n$. (a) $n = 1$, (b) $n = 10$, (c) $n = 100$ and (d) $n = \infty$. 

displacements induced by such residual are

\[ u = -\int_\Omega G(x_1 - \xi_1, x_2 - \xi_2) r(\xi_1, \xi_2) \, d\xi_1 \, d\xi_2. \tag{10} \]

As \( n \to \infty \), the effect of the local residual is felt with unwaning intensity at increasingly distant points along the emerging characteristics. Thus, local residuals have the effect of polluting the solution out to infinity. This precludes a posteriori estimation of local errors from local residuals. Indeed, imagine a remote element lying on a characteristic emanating from \( \Omega \). Such an element would be surrounded by an arbitrarily large region of vanishing residual and, thus, would have no local means of appreciating the large impending effect of the residual at \( \Omega \).

A class of adaptive meshing methods which does not rely on elliptic estimates consists of equi-distributing some suitable norm of the solution over the elements in the mesh. The precise choice of norm depends on the nature of the equations under consideration. Assume that the problem admits well-behaved solutions in some Banach space \( V \). Let \( | \cdot | \) denote a bounded semi-norm in \( V \). For instance, an appropriate choice of \( V \) for linear elasticity consists of the Sobolev space \( H^1 \) equipped with the energy semi-norm. Imagine now that the domain of the problem is discretized into a mesh \( \mathcal{M}_h \) containing elements \( \{ \Omega_e^h \mid e = 1, \ldots, \text{NUMEL} \} \). Let \( u_h^e \) denote a finite element approximation to \( u \), and let \( u_h^e \) and \( u^e \) denote their restrictions to element \( \Omega_e^h \), respectively. Then the mesh \( \mathcal{M}_h \) is taken to be optimal if \( |u_h^e| \) is the same for all elements in the mesh.

In some notable instances, the norm equi-distribution strategy can be derived from the interpolation error method of Diaz et al. [3]. This method is based on interpolation error bounds in Sobolev spaces. The essence of these bounds is as follows. Consider a generic element \( \Omega_e^h \) in the mesh and let \( u_h^e \) be the interpolant over \( \Omega_e^h \) formed from the nodal values of \( u^e \). Define the interpolation error \( E_h^e \) as some suitable norm of the difference \( u^e - u_h^e \). A typical bound of the interpolation error is then of the form

\[ E_h^e \leq C(h^e)^\alpha |u^e|, \tag{11} \]

where \( h^e \) is the diameter of \( \Omega_e^h \) and the semi-norm appearing on the right-hand side depends on the semi-norm used to define the interpolation error. For example, the following general class of bounds was derived by Ciarlet [15]. Let \( k \geq 0 \) and \( m \geq 0 \) be integers and \( p, q \in [1, \infty] \) be numbers such that \( W^{k+1,p}(\Omega_e^h) \) is contained in \( W^{m+1,\infty}(\Omega_e^h) \) with a continuous injection. Let the interpolation functions in \( \Omega_e^h \) include the complete set of polynomials up to order \( k \). Then there exists a constant \( C \) such that, for all \( u^e \in W^{k+1,p}(\Omega_e^h) \),

\[ |u^e - u_h^e|_{m,q,\Omega_e^h} \leq C[\text{meas}(\Omega_e^h)]^{1/q - 1/p} \left( \frac{h^e}{\rho} \right)^{k+1} |u^e|_{k+1,p,\Omega_e^h}, \tag{12} \]

where \( \text{meas}(\Omega_e^h) \) is the measure of \( \Omega_e^h \) and \( \rho^e \) the diameter of the largest ball contained in \( \Omega_e^h \). The requisite continuous injection condition is satisfied if [15]

\[ \frac{1}{q} = \frac{1}{p} - \frac{k+1-m}{n}, \quad k+1-m < \frac{n}{p}, \tag{13} \]

where \( n \) is the dimension of \( \Omega_e^h \), taking \( q \leq 1 \).

If the final result is replaced by the condition for \( q \) fixed, the solution is obtained.

It bears on the theory and practice of norms, for the class of estimates solutions such that the condition a has been satisfied.

Because of the mathematical instabilities in the velocity field near to a point, consistent with the problem interval \( x = 0 \) are prescribed by downslopes conditions for all \( x \) relative to a vertical tangent at the point.
where \( n \) is the number of spatial dimensions. These requirements are satisfied by, for instance, taking \( q = p = 2 \) and \( k + 1 = m \), which is the case considered by Diaz et al. [3].

If the finite element approximation interpolates well the exact solution, \(|u^*|\) in (11) can be replaced by \(|u_h^*|\). A global measure of the interpolation error is furnished by \( E_h = \sum E_h^* \). A minor extension of a proof by Devloo et al. [4] shows that the mesh which minimizes \( E_h \) for a fixed number of elements NUMEL is that for which \((h^*)^\alpha |u_h^*|\) takes a uniform value on all elements. If, for instance, the interpolation error is measured in terms of the energy norm, which corresponds to taking \( q = p = 2 \) and \( k = 0 \), \( m = 1 \) in (12), then \( \alpha = 0 \) and the optimality condition amounts to equi-distributing the energy norm. This method of mesh adaptivity has been successfully used in the context of linear elasticity [16, 17].

It bears emphasis that the interpolation error bounds stem directly from interpolation theory and are related to specific boundary value problems only as regards the proper choice of norms. This choice depends critically on the nature of the admissible space of solutions \( V \) for the class of problems under consideration, since, evidently, the norm chosen for equi-distribution must be bounded in \( V \). It is interesting to note that, in essence, interpolation error estimates simply provide an indicator of when the mesh is too coarse to adequately resolve the solution. In this sense, methods based on the minimization of interpolation error may be thought of as being ‘resolution driven’, in that the adaptive meshing strategy strives to eliminate elements where the variation of the solution is too high for the element interpolation to adequately resolve the solution.

Because the interpolation error method and associated norm equi-distribution strategies do not rely on the ellipticity of the equations, they are well-suited to the analysis of shear banding instabilities. The central question concerns the proper choice of norms. Although a rigorous mathematical analysis is lacking, it appears that the boundedness of the variation of the velocity field is assured under rather general, even pathological, conditions. To illustrate this point, consider the problem of a layer of width \( 2a \) undergoing simple shear. The solution to this problem is one-dimensional. We take the domain of definition of the solution to be the interval \( x \in [-a, a] \). Assume that the solid obeys a shear stress–shear strain relation characterized by an initial ascending portion, a maximum at some critical strain, and a subsequent downsloping portion where the shear stress \( \tau \to 0 \) as \( \gamma \to \infty \). Assume that velocity boundary conditions of the type

\[
v(-a, t) = -V, \quad v(a, t) = V, \quad t \geq 0
\]

are prescribed on the boundary of the layer. Evidently, this problem admits the family of solutions

\[
\tau(x) = 0, \quad -a \leq x \leq a, \quad v(x) = -V, \quad -a \leq x < c, \quad v(x) = V, \quad c < x \leq a
\]

for all \(-a < c < a\). These solutions correspond to two blocks \([-a, c], [c, a] \) sliding rigidly relative to each other at zero stress. The rate of deformation is

\[
l(x) = v'(x) = 2V\delta(x - c),
\]

where the derivative is defined in the distributional sense. It follows that the derivative of \( v(x) \)
defines a Borel measure and, hence, \( v(x) \) is of bounded variation. Indeed, the variation of \( v(x) \) is \( 2V \). It is interesting to note that the \( H^s \) norms of \( v(x) \), \( s \geq 0 \), are undefined. In actual reality, if all relevant physical mechanisms are properly accounted for, processes of localization are self-limiting and pathological solutions of this type do not arise. It is remarkable, however, that even in the singular limit represented by solutions (15) the variation of the velocity field remains bounded.

On a first inspection, the above example seems limited in that it is one-dimensional. However, well-developed shear bands are essentially one-dimensional on a local scale, i.e., the solution is rapidly varying normal to the band, and slowly varying in the direction of the band, with only one main velocity component in the direction of the polarization vector. Thus, the above example represents generic behavior of shear band solutions in any number of spatial dimensions. Based on these considerations, it appears plausible that BV is indeed an appropriate admissible solution space for problems involving localization. In fact, the space BV has become central in the analysis of shock waves and discontinuous phenomena in chemical physics, both phenomena not unrelated to localization in solids.

While a general mathematical theory of localization is yet to emerge, the special case of ideal plasticity has been extensively investigated by Mathies et al. [18] and Mathies [19]. This case is of interest here because the governing equations are on the elliptic-hyperbolic boundary. Mathies et al. [18] and Mathies [19] have shown that the solutions to the ideally-plastic problem lie in the space \( \text{BD} \) of functions of bounded deformation. This space reduces to the conventional space \( \text{BV} \) of functions of bounded variation in one dimension. In general, \( \text{BD} \) extends both \( W^{1,1} \) and \( \text{BV} \) [18]. In fact, based on a prior counter-example by Orstein [20], Mathies et al. [18] have given an example of a velocity field which is of bounded deformation but not of bounded variation. However, as previously remarked, well-developed shear bands, unlike ideally-plastic solutions, are essentially one-dimensional on a local scale. Indeed, the counter-example of Orstein [20] and Mathies et al. [18] bears little discernible resemblance to a shear band solution. In view of the locally one-dimensional character of shear bands, and recalling that \( \text{BD} \) reduces to \( \text{BV} \) in one dimension, it does not appear necessary to enlarge the admissible space beyond \( \text{BV} \) in the present context.

The fact that boundedness of the variation of localized velocity fields seems to be assured under rather general conditions suggests adopting a mesh adaptation strategy based on equi-distribution of variation. Thus, a mesh will be presumed optimal if the variation \( |v|_{\text{BV},[a,b]} \) of the velocity field is uniform over the elements of the mesh. In one dimension, the variation of \( v(x) \) over an interval \([a, b]\) is defined as

\[
|v|_{\text{BV},[a,b]} = \sup \sum_{k=1}^{N} |v(x_k) - v(x_{k-1})|, \tag{17}
\]

where the supremum is taken over all sequences \( a = x_0 \leq x_1 \leq \ldots \leq x_N = b \). If, in addition to being of bounded variation, the function \( v(x) \) is in \( W^{1,1}([a, b]) \), then (17) can be expressed as

\[
|v|_{\text{BV},[a,b]} = \int_a^b |v'(x)| \, dx, \tag{18}
\]

which is the variation of a function defined on sample points containing

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which is the $W^{1,1}$ semi-norm. In multiple dimensions, several definitions of functional variation have been proposed [21]. Here we have adopted Arzelà's definition [21]. Let $f$ be a function defined on $\Omega \subset \mathbb{R}^n$ and $\Gamma = \{x_{i0} \leq x_{i1} \leq \cdots \leq x_{in}\}_{i=1}^n$ be any nondecreasing set of points contained in $\Omega$. The variation of $f$ over $\Omega$ is then defined as

$$|f|_{BV, \Omega} = \sup_{\Gamma} \sum_{k=1}^{N} |f(x_k) - f(x_{k-1})|,$$

(19)

where the supremum is taken over all nondecreasing paths $\Gamma$ contained in $\Omega$. For a vector valued function such as $\mathbf{v}$, the definition of variation can be extended as

$$|\mathbf{v}|_{BV, \Omega} = \max_i |v_i|_{BV, \Omega}.$$

(20)

Finally, in computing the variation of $\mathbf{v}_h$ over an element $\Omega_h^e$, the following simple expression suffices:

$$|\mathbf{v}_h^e|_{BV, \Omega_h^e} = \max_i \left( \max_{a,b} |v_{ia}^e - v_{ib}^e| \right),$$

(21)

where $v_{ia}^e$ are the nodal components of $\mathbf{v}_h^e$ and $a, b$ range over the nodes of the element.

Our adaption criterion may now be simply formulated as the requirement that indicator (21) be uniformly distributed over the mesh at all times. In enforcing this criterion, we introduce a tolerance $\text{VARMAX}$ and target an element $e$ for refinement when

$$|\mathbf{v}_h^e|_{BV, \Omega_h^e} \geq \text{VARMAX}.$$ 

(22)

As stated, this criterion presumes that the variation of the solution is initially uniform or nearly uniform over all elements of the mesh, and that the elements in the mesh are initially all identical or nearly identical. This condition is met, for instance, if localization takes place from a uniform state interpolated by a uniform mesh. Under more general conditions, criterion (22) is to be applied locally over a patch of initially nearly identical elements where the solution is initially of nearly uniform variation, and where subsequent localization is anticipated. Alternatively, the element variations considered in the adaption criterion may be weighted by their initial values. As the solution in the patch deviates from the locally uniform state and progresses into an increasingly localized mode, condition (22) will be activated within the evolving shear band and refinement will be triggered. The meshing procedure employed in our numerical tests is described in Section 5.

It is not known to us whether the present adaption criterion can be derived from interpolation error bounds of the type discussed above. The question is whether a norm can be identified such that the corresponding interpolation error is bounded as

$$E_h^e \leq C |u|_{BV, \Omega_h^e}.$$

(23)

In one dimension, if the solution $u(x)$ lies in $W^{1,1}$, then the variation of the solution coincides with the $W^{1,1}$ semi-norm (18), and the general bound (12) can be specialized with $q = \infty$. 

addition to
\( m = 0, \, k = 0, \, p = 1 \) to give
\[
|u^e - u_h^e|_{L^\infty(\Omega_h^e)} \leq C|u|_{BV, \Omega_h^e} .
\] (24)

Under these conditions, the equi-distribution of variation strategy minimizes the \( L^\infty \) interpolation error. It would be of interest to know whether bounds of this type extend to general functions of bounded variation and to multiple dimensions.

4. Consistent transfer operators

In this section we concern ourselves with the problem of formulating finite element solutions for history-dependent materials on an evolving mesh. In contrast to linear elastic solids, for which the displacements can be recalculated readily as successive refinements are introduced, the history-dependent nature of plastic solids necessitates the transfer of all nodal and state variables from mesh to mesh. Several questions arise in this regard, namely, (i) how to effect the transfer of state variables in a manner consistent with the constitutive algorithm, (ii) how to integrate the state transfer into the equilibrium iterations, (iii) how to make the state transfer compatible with the displacement field on the new mesh, and (iv) how to minimize numerical diffusion of the state fields. Next, we show how the transfer operator can be consistently derived from an extended variational principle in a manner which addresses these issues.

For the formulation of finite element procedures it proves convenient to rephrase (1) in weak form, leading to the integral statements
\[
\int_{B_0} P : \nabla \phi \, dV_0 - \int_{B_0} \rho_0 B : \delta \phi \, dV_0 - \int_{F_{w0}} \bar{T} : \delta \phi \, dS_0 = 0 ,
\]
\[
- \int_{B_0} (F - \nabla_0 \phi) : \delta P \, dV_0 - \int_{F_{w0}} (\phi - \bar{\phi}) \cdot \delta P \cdot N \, dS_0 = 0 ,
\] (25)

where \( \delta \phi \) and \( \delta P \) are arbitrary variations of the deformation mapping and stress field, respectively, \( \nabla_0 \) denotes the gradient operator over \( B_0 \), \( dS_0 \) and \( dV_0 \) are the elements of undeformed area and volume, respectively, and, for simplicity, we consider the quasistatic case and neglect the inertia forces. In (25) and subsequently, the symbol \( ' : ' \) is used to denote the inner product between second order tensors, e.g., \( A : B = A_{ij} B_{ij} \), where the summation convention on repeated indices is implied.

For a plastically deforming solid, the constitutive response is path dependent. Envision, therefore, a process of step-by-step integration of the governing equations in which the body forces \( B \), prescribed boundary tractions \( \bar{T} \) and prescribed boundary displacements \( \bar{\phi} \) are specified at times \( t_0, t_1, \ldots, t_n, t_{n+1} = t_n + \Delta t, \ldots \). Let \( \phi_n, F_n, P_n, F^p_n \) and \( Q_n \) denote the values of the deformation mapping, deformation gradients, first Piola–Kirchhoff stress tensor, plastic part of the deformation gradients and internal variables at time \( t_n \), respectively. For simplicity of notation, we define a state array \( \Lambda = \{ P, F^p, Q \} \) encompassing all state variables.

We shall seek to satisfy (25) at the discrete times \( t_n \). In particular, for time \( t_{n+1} \) one has
\[ \int_{B_0} P_{n+1} : \nabla \phi \, dV_0 - \int_{B_0} \rho_0 B_{n+1} \cdot \delta \phi \, dV_0 - \int_{\Gamma_0} T_{n+1} \cdot \delta \phi \, dS_0 = 0 , \]

\[ - \int_{B_0} (F_{n+1} - \nabla \phi_{n+1}) : \delta P \, dV_0 - \int_{\Gamma_0} (\phi_{n+1} - \bar{\phi}_{n+1}) \cdot \delta P \cdot N \, dS_0 = 0 . \]  

(26)

In addition, we shall assume that a state update algorithm has been defined, giving the updated state \( \Lambda_{n+1} \) as a function of the initial conditions \( \Lambda_n \), the updated value of the deformations gradients \( F_{n+1} \) and \( \Delta t \). In particular, we write

\[ P_{n+1} = \hat{P}(F_{n+1}; \Lambda_n, \Delta t) . \]  

(27)

As in the case of equilibrium and compatibility, we shall endeavor to satisfy (27) in its weak form

\[ \int_{B_0} [\hat{P}(F_{n+1}; \Lambda_n, \Delta t) - P_{n+1}] : \delta F \, dV_0 = 0 . \]  

(28)

In the simple case of finite elasticity, (28) reduces to

\[ \int_{B_0} [\frac{\delta W}{\delta F} (F_{n+1}) - P_{n+1}] : \delta F \, dV_0 = 0 , \]  

(29)

where \( W(F) \) is the strain energy potential of the solid. Then, the weak forms (26), (28) of all the field equations can be jointly derived as the Euler equations of the Hu–Washizu functional

\[ L(\phi, F, P) = \int_{B_0} [W(F) - P : (F - \nabla \phi)] \, dV_0 - \int_{B_0} \rho_0 B \cdot \phi \, dV_0 - \int_{\Gamma_0} T \cdot \phi \, dS_0 \]

\[ - \int_{\Gamma_0} (\phi - \bar{\phi}) \cdot P \cdot N \, dS_0 , \]  

(30)

Equations (26) and (28) are now taken as a basis for consistently deriving the finite element equations in the presence of mesh refinement. To this end, assume that \( B_0 \) is partitioned into finite element meshes \( M_n \) at times \( t_n \). For a given mesh, the following spatial interpolation is introduced for the deformation mapping and the state variables:

\[ \phi(X) = \sum_a x_a N_a(X) , \quad \Lambda(X) = \sum_e \sum_p \Lambda^e_p M^e_p(X) , \]  

(31)

where \( N_a \) are the conforming interpolation functions for the deformation mapping, \( M^e_p \) are the element interpolation functions for the state variables, \( x_a \) the spatial coordinates of the nodes and \( \Lambda^e_p \) are the state degrees of freedom for element \( e \). Because the state variables enter (26) and (28) undifferentiated, the shape functions \( M^e_p \) need not be continuous across element boundaries. The displacement shape functions \( N_a \), on the other hand, are subject to the usual requirement of \( C^0 \)-continuity.
In subsequent discussions, we shall assume that the displacement boundary conditions (1b) are identically satisfied over \( \Gamma_{uo} \), so that the second term in (26b) vanishes identically. Substitution of (31) into (26) and (28) gives the set of discretized equations

\[
\sum\ell L^e_{n+1} p^e_{n+1} - f_{n+1} = 0, \quad M^e_{n+1} f^e_{n+1} - L^e_{n+1} x_{n+1} = 0, \quad \dot{P}^e(F^e_{n+1}; A^e_{n}, \Delta t) - M^e_{n+1} p^e_{n+1} = 0, \quad (32)
\]

where one writes

\[
\begin{align*}
L^e_{iipj} &= \int_{\Omega_0} \delta_{ij} N_{b_1} M_p dV_0, \\
F_{i} &= \sum\ell \left[ \int_{\Omega_0} \rho_0 B_i N_a dV_0 + \int_{\Gamma_0} \bar{T} i N_a dS_0 \right], \\
M^e_{iipj} &= \int_{\Omega_0} \delta_{ij} \delta_{ip} M^e_p M^e_q dV_0, \\
\dot{P}^e_{iip}(F^e; A^e_{n}, \Delta t) &= \int_{\Omega_0} \dot{P}^e_{ij}(F^e; A^e_{n}, \Delta t) M^e_p dV_0.
\end{align*}
\]

In writing (32) in matrix form, the groups of indices of the type \( \{ia\} \) and \( \{iip\} \) have been redefined as single indices in the usual way. Thus, \( L^e_{iipq} \) and \( M^e_{iipj} \) are mapped into matrices \( L^e \) and \( M^e \), respectively, and \( f_{ia} \), \( F^e_{i} \) and \( P^e_{i} \) into linear arrays denoted \( f \), \( F^e \) and \( P^e \), respectively. The superindex \( e \) in the latter arrays differentiates them from the symbols \( F \) and \( P \) employed for the deformation gradients and the stress field, respectively. The subindices \( n \) and \( n + 1 \) in (32) refer to the time at which the various arrays are sampled. Arrays such as \( L^e \) vary from \( t_n \) to \( t_{n+1} \) as a consequence of the change of mesh from \( M_n \) to \( M_{n+1} \). It should be carefully noted that in (32c), the initial state \( A^e_{n} \) is interpolated on mesh \( M_n \), whereas all the remaining variables are interpolated on mesh \( M_{n+1} \).

Equations (32b) and (32c) are decoupled on an element-by-element basis. These equations can be simplified further by recourse to a lumping procedure. To this end, we start by computing all integrals in (33) by numerical quadrature based on element quadrature points and weights \( \{ x^e_{p} \} \) and \( \{ w^e_{p} \} \), respectively. Next, we normalize the element shape functions \( M^e_p \) to satisfy

\[
M^e_p(\xi^e_q) = \delta_{pq}, \quad (34)
\]

Then, the parameters \( A^e_p \) in (31b) take the significance of being the values of the state variables at the quadrature points within the elements. Furthermore, (32b) and (32c) decouple on a quadrature point by quadrature point basis to become

\[
F^e_{n+1} = L^e_{n+1} x_{n+1}, \quad P^e_{n+1} = \hat{P}^e(F^e_{n+1}; T A^e_{n}, \Delta t), \quad (35)
\]

where the lumped \( L^e \)-array is redefined as

\[
L^e_{iipj} = \delta_{ij} N_{b_1}(\xi^e_p), \quad (36)
\]

and the state 'transfer operator' \( T \) between meshes \( M_n \) and \( M_{n+1} \) is defined as

\[
(TA^e_{n})_p = \sum_q A^e_{q,n} M^e_{q,n}(\xi^e_{p,n+1}). \quad (37)
\]
Equation (35a) simply states that the deformation gradients at the quadrature points are computed by direct differentiation of the displacement field. Likewise, in the case of an elastic solid for which the dependence of \( \mathbf{P}_{n+1} \) on the initial conditions (\( \mathbf{A}_n \), \( \Delta t \)) is lost, (35b) simply states that the stresses at the quadrature points are computed directly from the corresponding deformation gradients by means of the stress-strain relations.

By contrast, in the case of a history-dependent solid, (36) and (37) convey non-trivial information and constitute the central result of this section. Thus, (36) and (37) provide a precise rule for updating the state variables in the presence of mesh refinement. The content of (36) and (37) may be stated by saying that, to effect a state update between two different meshes, the initial state fields \( \mathbf{A}_n(X) = \sum_q A_{q,n}^{(e)} M_{q,n}^{e}(X) \) must be first 'transferred' to the new mesh by sampling them at the new quadrature points \( \xi_{p,n+1}^e \). This defines the transfer operator \( T \). Then, a regular state update is effected at the quadrature points of the new mesh. It should be noted that this procedure requires handling two meshes throughout the calculations: the initial mesh \( M_n \) for the time step, which is required to define \( \mathbf{A}_n(X) \), and the updated mesh \( M_{n+1} \). Only when the updated solution has been computed can \( M_n \) be discarded.

Equations (36) and (37) also show how to integrate the transfer operator into an equilibrium iteration. To see this, we start by noting that (32a) and (35) can be combined to give a single system of nonlinear equations for \( x_{n+1} \) of the form

\[
\sum_{q} \int_{\Omega_0} L_{q,n+1}^{e} \Phi(L_{q,n+1}^{e}, x_{n+1}, T\mathbf{A}_n^{e}, \Delta t) \, dV_0 - f_{n+1} = 0. \tag{38}
\]

The final mesh \( M_{n+1} \) on which the solution \( x_{n+1} \) is defined is unknown beforehand. For each pair of meshes \( M_n \) and \( M_{n+1} \), let us arbitrarily define the mesh transfer operator \( T \) for displacements as

\[
(Tx)_i = \sum_b x_{ib,n} N_b(x_{a,n+1}). \tag{39}
\]

Thus, to obtain the nodal displacements of \( T \mathbf{x}_n \) in \( M_{n+1} \), one simply samples the interpolated deformation mapping \( \Phi_n(X) = \sum_a x_a N_{a,n}(X) \) on \( M_n \) at the nodal points of \( M_{n+1} \). The precise details of this operation are not critical since it is only used to define an initial guess for the equilibrium iteration. If the updated solution is unique and the iteration converges, the results should be independent of the initial guess, by virtue of the fact that the state updates are always performed from a fixed set of initial conditions \( \mathbf{A}_n \). We shall employ the symbol \( T \) for all transfer operators, which will be identified by their arguments.

The final solution \( x_{n+1}, M_{n+1} \) is obtained by means of two nested iterations. The outer iteration concerns the determination of the mesh \( M_{n+1} \), which varies according to some adaption criterion. The inner loop concerns the equilibrium iteration at fixed mesh. We set \( M_{n+1}^{(0)} = M_n \). For each iteration \( M_{n+1}^{(k)} \), we set the initial guess for the equilibrium iteration equal to \( T \mathbf{x}_n \), where \( T \) is the displacement transfer operator between \( M_n \) and \( M_{n+1}^{(k)} \). An equilibrium iteration, e.g., a Newton–Raphson iteration, then determines a converged solution \( x_{n+1}^{(k)} \). The iteration stops when the mesh adaption criterion is satisfied for the converged solution.

The above procedure generalizes Simo and Taylor’s [22] consistent solution procedure for plasticity to the case of a varying mesh. The need to transfer state variables between meshes places taxing demands on the accuracy of the state variable interpolation within the elements.
If, as tends to be the case with the four-node isoparametric element, extrapolation from the quadrature points defines a jagged variation of the state fields across elements, the newly refined mesh picks up the noise and the solution becomes corrupted. We have based all our calculations on six-noded triangular elements with three quadrature points per element, resulting in a linear representation of the state variables within the element. The fact that the displacements are quadratic and the strains are linear within the element seems to minimize the amount of noise in the state variable fields. The elements are quite inexpensive since, in exchange for quadratic nodal interpolation, they only require three quadrature points per element. In addition, triangular meshes can be conveniently constructed by a triangulation algorithm. These desirable properties render the six-noded element particularly attractive for adaptive meshing calculations involving history-dependent solids.

5. Meshing technique

We construct all meshes by Delaunay triangulation. The triangulation is based on the corner nodes of the elements, with the midnodes added subsequently. The precise sequence followed in effecting a mesh refinement is sketched in Fig. 2. Imagine that the central element in Fig. 2(a) is targeted for refinement. The initial mesh is constructed by Delaunay triangulation based on the corner nodes of the elements, shown in Fig. 2(b). To define the refined mesh, new corner nodes are added at the midides of the target elements. The element connectivity is then redefined by a Delaunay triangulation based on the new set of corner nodes. Finally, the midside special care outside the element algorithm is $O(N^2)$. Hence, triangulation. Consequently, we adopted Crouzet's method of redefining the bandwidth.

6. Numeric

Next we present a few examples concerning one case, that of the remaining one being non-associative. We assume that local conditions are challenging, shear band delay or even shear bands and shear failure occur.

6.1. Plane

Our first example is the $x_1-x_2$ plane subjected to shear and boundary conditions $T = 1$ and $V = 1$.

The plastic multiplier is defined as

$\tilde{T}$

where $T$ is the input stress and $V$ is the damage. The solid is assumed to have dissipation and power.
the midside nodes are added, with the result shown in Fig. 2(d). For non-convex domains, special care has to be exercised to eliminate elements resulting from the triangulation lying outside the domain of analysis. Sloan [5] has given a particularly efficient Delaunay triangulation algorithm which we have adopted in our calculations. The operation count for Sloan’s algorithm is \( O(N^{5/4}) \) on the average, as \( N \to \infty \). In the worst case, the operation count is \( O(N^2) \). Here \( N \) denotes the number of nodes in the mesh. A difficulty with meshes defined by triangulation, however, is that their bandwidths tend to become inordinately large. Consequently, we have combined the triangulation algorithm with a bandwidth minimizer. We have adopted Crane et al.’s [27] implementation of Gibbs et al.’s algorithm [28], which reduces the bandwidth and profile of a sparse matrix by a sequence of row and column permutations.

6. Numerical examples

Next we consider two examples which illustrate the performance of the method. Both examples concern localizations instabilities in a rectangular solid deforming in plane strain. In one case, the solid obeys \( J_2 \)-flow theory and softens after an initial hardening state. In the remaining example the solid obeys a Drucker-like pressure-sensitive yield criterion with a non-associated flow potential, and exhibits non-softening stress–strain behavior. In both cases, the assumed constitutive behavior results in localization instabilities when the appropriate local conditions (e.g., eq. (5)) are met. Several aspects of the calculations are especially challenging. Indeed, conventional finite element methods have difficulty producing sharp shear bands at arbitrary angles to the mesh. In addition, conventional elements frequently delay or even suppress localization, and tend to be overly stiff in the post-localization regime. The numerical examples that follow demonstrate the ability of the present method to overcome these difficulties, as well as to resolve multiple evolving scales simultaneously.

6.1. Plane strain tension, \( J_2 \)-flow theory

Our first example is concerned with the development of shear bands in a rectangular solid subjected to plane-strain uniaxial tension. The solid occupies the domain \([-a, a] \times [-b, b]\) in the \( x_1-x_2 \) plane, and is loaded by prescribing the axial velocity at \( x_2 = \pm b \). The complete set of boundary conditions adopted in the calculations is

\[
T_1 = 0, \quad T_2 = 0, \quad x_1 = \pm a, \\
T_1 = 0, \quad V_2 = \pm \dot{V}, \quad x_2 = \pm b,
\]

where \( T \) is the traction vector per unit undeformed area, \( V \) is the Lagrangian velocity field, and \( \dot{V} \) is the prescribed velocity.

The plastic flow of the solid is modeled within the framework of finite deformation multiplicative plasticity. Thus, we assume that the deformation gradients admit a decomposition of the form \( \mathbf{F} = \mathbf{F}^e \mathbf{F}^p \), where \( \mathbf{F}^e \) and \( \mathbf{F}^p \) are the elastic and plastic parts of \( \mathbf{F} \), respectively. The solid is assumed to obey a standard \( J_2 \)-flow theory of plasticity with isotropic hardening and power rate-sensitivity. The hardening law giving the flow stress as a function of an
effective plastic strain comprises two terms: a power hardening term which dominates during the first stages of plastic deformation; and a negative quadratic term which takes over at large strains and which introduces softening into the plastic response. The elastic relations are assumed to be linear and isotropic in terms of the elastic Lagrangian strains. The nearly incompressible character of plastic flow is handled by an assumed strain method in which the volumetric deformations are evaluated at the centroid and taken to be uniform over the element. Details of the formulation may be found in [23].

The state variables are updated by means of a fully implicit backward Euler scheme, with the consistent tangents computed by numerical differentiation. The global equilibrium equations are solved by a Newton–Raphson iteration combined with an arc-length method. This consists of prescribing the incremental value of the fastest growing degree of freedom. This method was originally proposed by Tvergaard et al. [24], and bears an interesting relation to localization. If the specimen is loaded by displacement control, by the maximum principle the fastest growing degree of freedom must necessarily lie on the boundary for as long as the equations remain elliptic. Therefore, in this range the arc-length method reduces to conventional displacement control. However, as localization sets in, ellipticity is lost and the fastest growing degree of freedom is found in the interior of the solid. Thus, the migration of the active constraint from the boundary to the interior signals the onset of localization.

The values of the material parameters adopted in the calculations are $E/\sigma_y = 500$, $\nu = 0.3$, $m = 200$, $n = 5$, where $E$ and $\nu$ are the elastic moduli, $\sigma_y$ is the initial yield stress, $m$ is the rate sensitivity exponent, and $n$ is the initial hardening exponent. The resulting stress–strain law in plane-strain uniaxial tension is shown in Fig. 3. The aspect ratio of the specimen is $b/a = 1.5$. An imperfection in the form of a bell-shaped initial distribution of effective plastic strain is introduced about the center of the specimen. The exact form of the imperfection is

$$\bar{\varepsilon} = A \exp(-r^2/2c^2),$$

where $r$ is the radial distance to the origin. In calculations, the amplitude of the imperfection is assigned the value $A = 0.001$, whereas the thickness of the imperfection is $c/a = 0.25$. The initial two-fold symmetry of the solution is preserved at all subsequent times. This enables the computations to be confined to one quarter of the specimen. The initial mesh is shown in Fig. 4(a).

![Fig. 3. Homogeneous and localized force-elongation curves for rectangular solid subjected to plane-strain tension.](image-url)
Meshes and distributions of effective plastic strain at various stages during the solution are shown in Fig. 4. Initially, the variation of the velocity field is ostensibly the same over all elements and equals $\dot{V}/6$. The distribution of effective plastic strain is likewise nearly uniform. At some point beyond the peak load, the effective plastic strains begin to exhibit signs of localization, Fig. 4(b). When the maximum element variation of the velocity field reaches $0.9\dot{V}$, all elements with variations over $0.3\dot{V}$ are targeted for refinement. The meshes and effective plastic strain distributions before and after refinement are shown in Figs. 4(b) and 4(c). The smoothness of the effective plastic strain field afforded by the six-noded elements with linear state interpolation, as well as the accuracy of the state transfer operator, are evident from the figures. The refinement criterion is enforced repeatedly during the calculations, with the results shown in Figs. 4(d)–(i). The rapid rate of growth and increasingly localized nature of the plastic deformations is clearly apparent from these figures. The corresponding force elongation curve is shown in Fig. 3. As may be seen, all the successive

\begin{equation}
(41)
\end{equation}

perfection is $= 0.25$. The enables the own in Fig.

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{fig4}
\caption{Successive refined meshes and distributions of effective plastic strain for plane-strain tension problem. (b), (d), (f) and (h) show solution before refinement. (c), (e), (g) and (i) show mesh after refinement and effective plastic strains after an application of transfer operator.}
\end{figure}
6.2. Plane-stress

Next we consider a plane-stress solid occupying $x = \pm b$. The stress is given by $\sigma = -\frac{V}{b^2}$. As in the framework of the Drucker-Prager criterion, the model is characterized by the stress-strain relationship. The elastic response is described by a linear elastic law. The model is also characterized by the lack of normal hardening and geometrical nonlinearity due to compression.

The values are $\phi_0 = 20^\circ$, $\phi = ...$

Fig. 4. Continued.

Fig. 5. Mesh after four adapt refinements and the uniform refinement. The mesh reveals...
6.2. Plane–strain compression of a sand specimen

Next we consider the case of a rectangular sand specimen constrained to deform in plane–strain in the $x_1-x_2$ plane and subjected to axial compression in the $x_1$ direction. The solid occupies the domain $[-a, a] \times [-b, b]$, and the axial velocities are prescribed at $x_2 = \pm b$. The boundary conditions appropriate to this problem are as in (40), with $\dot{V}$ replaced by $-\ddot{V}$. As in the preceding example, the constitutive behavior of the solid is modeled within the framework of finite deformation multiplicative plasticity. The solid is assumed to obey a Drucker–Prager yield criterion with a non-associated flow rule. The plastic potential is chosen of the same form as the yield criterion, with a tension cap in hydrostatic tension. Hardening is modeled through a variation of the friction angle, which steadily increases from an initial to a saturation value. The corresponding rate-independent stress–strain curve in plane-strain compression is depicted in Fig. 6. It should be noted that no softening is introduced in the plastic response. Consequently, in the present example localization is driven strictly by the lack of normality of the solid. The slight apparent hardening in the saturated range, Fig. 6, is a geometrical effect associated with the increase in cross section of the specimen during compression. A slight rate-dependency is introduced by means of a linear overstress model. The elastic relations are assumed to be linear and isotropic in terms of the elastic Lagrangian strains. The details of the solution procedure are as in the preceding example.

The values of the material parameters adopted in the calculations are $E/p_0 = 500$, $\nu = 0.22$, $\phi_0 = 20^\circ$, $\phi_s = 30^\circ$, $\psi = 0^\circ$, where $E$ and $\nu$ are the elastic moduli, $p_0$ is the cohesive pressure, $\phi_0$
is the initial friction angle, $\phi_0$ is the saturation friction angle, and $\psi$ is the dilatancy angle. The aspect ratio of the specimen is $b/a = 2$. An imperfection in the form of a uniform initial distribution of effective plastic strain with a bell-shaped well about the origin is introduced in the solid. The exact form of the imperfection is

$$\bar{\varepsilon} = A[1 - \exp(-r^2/2c^2)].$$  \hspace{1cm} (42)

The amplitude of the imperfection is assigned the value $A = 0.001$, whereas the thickness of the imperfection is chosen to be $c/a = 0.25$. As before, the initial symmetry of the solution is preserved throughout and the calculations are restricted to one quarter of the specimen. The initial mesh is shown in Fig. 7(a).

In the initial stages of the solution, the variation of the velocity field is ostensibly uniform over all elements and equals $V/8$. The distribution of effective plastic strain is likewise nearly uniform. Mesh refinements are introduced at regular intervals when the maximum element variation of the velocity field reaches the value $0.9 V$. Then, the elements refined are those for which the velocity variation exceeds $0.25 \bar{V}$. The meshes resulting from the first few adaptions are shown in Figs. 7(b)–(e). Again, the catastrophic rate of growth and increasingly localized character of the effective plastic strains are evident from the figures. The corresponding force-elongation curve is shown in Fig. 6, and is seen to cover the early stages of the post-localization regime. In the present example, some degree of ‘kickback’ is observed in the force-elongation curve.

Unlike the preceding case of a solid obeying $J_2$-flow theory, in which the developing shear bands are relatively featureless in the direction of the band, the shear bands in the frictional solid exhibit a good deal of structure. As soon as the bands are sufficiently developed, two well-differentiated segments become apparent: one in the interior of the solid, and another next to the free surface, see, e.g., Figs. 7(d) and 7(e). The interior segment contains higher levels of effective plastic strain than the boundary segments. Most significantly, there is a marked mismatch in the orientation of the interior and boundary segments of the shear bands. This misorientation is plainly manifest in the final mesh, Fig. 8.

An analysis given by Needleman and Ortiz [11] helps to explain these features of the solution. By an investigation of the complementing condition in pressure-sensitive frictional solids, Needleman and Ortiz [11] have determined the critical plastic modulus for localization at a free boundary to be

$$\frac{h_c}{\mu} = 0$$  \hspace{1cm} (43)

and the corresponding band orientations

$$\tan \theta = \pm \left( \frac{1 + \tan \psi}{1 - \tan \psi} \right)^{1/2}.$$  \hspace{1cm} (44)

In arriving at these results, geometrical effects have been neglected. By way of contrast, a standard bifurcation analysis yields [11]

$$\frac{h_c}{\mu} = \frac{\lambda + \mu}{4(\lambda + 2\mu)} (\tan \phi - \tan \psi)^2$$  \hspace{1cm} (45)

for the critical plastic modulus for localization in the interior of the solid, with band orientations.
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\[ (43) \]

\[ (44) \]

\[ (45) \]

Fig. 7. Successive refined meshes and distributions of effective plastic strain for plane-strain compression problem.
\[
\tan \phi = \pm \left[ \frac{1 + (\tan \phi + \tan \psi)/2}{1 - (\tan \phi + \tan \psi)/2} \right]^{1/2}
\]  \hspace{1cm} (46)

In (43) and (45), \( \lambda \) and \( \mu \) are the Lamé constants. It therefore follows that localization about traction-free boundaries, which necessarily occurs at peak load, is preceded by localization in the bulk, which may occur on the rising part of the stress-strain curve in solids lacking normality, as noted by Rudnicki and Rice [25]. The mismatch in orientation between the bulk bands and the boundary ‘kinks’ is evident from (44) and (46). For instance, for the values of the parameters employed in the calculations, it follows that \( h_c/\mu = 0.053 \) for localization in the bulk, and that the band orientations are \( \theta = \pm 53.4^\circ \) in the bulk and \( \theta = \pm 45.0^\circ \) at the boundary. The computed band orientations, as seen from Fig. 8, do indeed closely match these analytical values.

As noted by Needleman and Ortiz [11], the misorientation between bulk shear bands and boundary kinks profoundly affects the overall response of the solid. This mainly owes to the fact that the boundary kinks determine the relative velocities of the unloaded portions of the specimen, which essentially move as rigid blocks. Because these velocities are misaligned with the bulk bands, these must inevitably undergo dilatation in addition to shearing. This accounts for the increased levels of effective plastic strain which are observed in Figs. 7(d) and 7(e) within the interior segment of the bands. Moreover, in pressure-sensitive frictional materials of the type considered here, the dilatation imposed on the bulk bands can only occur for states of stress in the hydrostatic tension cap region. Consequently, the level of stress in the solid must drop to values of the order of the cohesion immediately following the formation of the boundary kinks. This accounts for the sudden reduction in bearing capacity of the specimen which is evident in the force-elongation curve, Fig. 6. Complete fixed-mesh solutions of the same problem given by Leroy and Ortiz [14] show that the load does indeed eventually stabilize at a level consistent with the cohesion of the material, in keeping with experimental observations [26].

Fig. 8. Mesh for the plane-strain compression problem after four adaptions.
Summary and conclusions

An adaptive meshing method tailored to problems of strain localization has been given and tested. Because the governing equations of interest here lose, or nearly lose, ellipticity at localization, methods of local error estimation based on elliptic estimates tend to break down. A simple example consisting of a solid deforming in antiplane shear and obeying $J_2$-deformation theory of plasticity with power-law hardening has been used to illustrate the manner in which local elliptic estimates break down as a result of the emergence of characteristic directions. The adaption strategy adopted consists of equi-distributing the variation of the velocity field over the elements of the mesh. A heuristic justification for the use of variations as indicators has been advanced, and possible connections with interpolation error bounds have been discussed. We have also addressed the problem of formulating implicit finite element solution procedures for history-dependent materials on an evolving mesh. We show that the Hu–Washizu principle can be used to determine a consistent mesh transfer operator for the state variables. The same variational principle provides guidelines for consistently formulating equilibrium iterations in the presence of an evolving mesh.

The appeal of adaptive meshing techniques in the context of localization instabilities, as well as in other problems exhibiting highly directional multiple scale features in the solution, stems mainly from the desire to accurately resolve the evolving shear bands, to free the calculations from any directional bias imposed by the mesh, and to eliminate locking effects resulting in an overly stiff post-localization response. The examples of application given demonstrate the usefulness of the present method in all these respects. For instance, the method is sensitive enough to detect localization in cases where the solid hardens at all times and the instability is driven by lack of normality, which is a far more exacting test than the case of a softening solid. The method also allows the bands to develop rapidly, with the attending abrupt drop (in fact kickback in one of the examples) in the force-elongation curve. The method has also proven successful at producing sharp shear bands misaligned with the principal mesh directions. All computed band orientations are consistent with theoretical predictions. The method is sensitive enough to reproduce the telltale kink bands characteristic of the effect of free boundaries on localization in pressure sensitive non-normal solids. The emergence of kink bands under these conditions was noted by Leroy and Ortiz [14] on the basis of a fixed mesh analysis. Again, the orientations of the kink bands are in agreement with recent analytical predictions [11].

Acknowledgment

The support of the Office of Naval Research through grant N00014-90-J-1758 is gratefully acknowledged.

References


1. Introduction

Localisation, including in the plastic strain of a perfect plastic and a perfectly plastic material, may be caused by the localization of a few small cracks. Because of the nature of the material and due to the different types of loading, the localization of cracks may be in the form of either a single crack or a multiple crack pattern. The rupture of a single crack is caused by the localization of a few small cracks. In the presence of localized stress--strain fields of a material, the crack may propagate in such a manner that it eventually becomes a fracture surface.