FINITE ELEMENT ANALYSIS OF TRANSIENT STRAIN LOCALIZATION PHENOMENA IN FRICTIONAL SOLIDS

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SUMMARY

Finite elements with embedded shocks are used to investigate transient strain localization phenomena in frictional solids. In particular, we seek to elucidate the effect of rate sensitivity and inertia on the development of shear bands in solids subjected to impulsive loading. As in the static case, our results show that shear banding may induce severe softening of the specimen even as the material steadily hardens. As expected, rate sensitivity retards the onset of structural softening and tends to stabilize the post-peak response. It is verified that the static solution is indeed recovered in the inviscid limit. Under dynamic conditions, shear bands are observed to propagate discontinuously, arresting and resuming propagation repeatedly before linking up with the boundary of the specimen. The direction of the band is equally unsteady. In addition, multiple shear banding, with the development of secondary and even tertiary bands, appears to be a prevalent mechanism at sufficiently high impact velocities.

INTRODUCTION

Bands of intense deformation, or shear bands, are a common occurrence in frictional materials: soils,\textsuperscript{1,2} rocks\textsuperscript{3} and concrete.\textsuperscript{4} In addition to constituting an important mechanism of inelastic deformation, shear bands often result in shear fractures and are, thus, coadjutant to failure. An example of the key role that shear banding may play in the response of frictional solids is furnished by the phenomenon of structural softening (Drescher and Vardoulakis\textsuperscript{5}, refer to this effect as geometrical softening). A post-bifurcation analysis of the plane strain compression test carried out by Leroy and Ortiz\textsuperscript{6} demonstrates that considerable softening in the overall response of the specimen may occur as a consequence of localization, even as the solid steadily hardens. Phenomena of this nature considerably compound the interpretation of experimental data and their application to constitutive identification (see, for instance, the review paper of Read and Hegemier\textsuperscript{7}).

The conditions under which localization instabilities develop are at present well-understood (see, for instance, the review paper of Rice\textsuperscript{8}). Frictional materials exhibit a number of peculiarities brought about by the non-associativity of plastic flow and the pressure sensitive character of yield. The fact that non-normality may promote localization instabilities was noted by Mandel\textsuperscript{9} and later conclusively established by Rudnicki and Rice.\textsuperscript{10} By contrast, a full understanding of the post-bifurcation behaviour of frictional solids is yet to emerge. The analyses carried out to date have been concerned with an infinite band in an unbounded medium.\textsuperscript{11,12} However, in situations such as those resulting in the aforementioned structural softening effect, elastic unloading, size effects and boundary conditions seem to play a significant role.\textsuperscript{6} Recent
developments in finite element methodology \(13-17\) have made possible a reliable analysis of strain localization under realistic conditions.

In this paper we seek to elucidate the effect of rate dependency and inertia on processes of strain localization in frictional solids. Rate dependency of the plastic flow has notable implications on localization phenomena which have been long recognized.\(18-20\) For instance, rate sensitivity is found to retard the inception and subsequent progression of shear banding. Recent experimental observations of Marchand and Duffy\(21\) on steel thin-walled cylinders subjected to high rates of loading lend support to these theoretical observations. The mechanisms at work here are identical to those responsible for similar delays occurring in the necking of a bar under uniaxial tension.\(22,23\) Another far-reaching consequence of rate dependency is the fact that it causes the field equations to retain their elliptic character at all times.\(24\) Under quasistatic conditions, and unlike the rate-independent case, uniqueness of the solution is not lost and the shear band thickness is set by the defect size. In contrast to rate sensitivity in metals, the specific case of frictional materials remains relatively unexplored. For instance, the question of how rate dependency affects structural softening appears not to have been addressed in the past.

There is an even more extreme paucity of results on the dynamic propagation of shear bands under multiaxial conditions. The problem has been studied by Freund \textit{et al.},\(25\) who considered the problem of a softening, rate-independent hyperelastic solid deforming in antiplane shear. A noteworthy outcome of their analysis is the ability of the band front to arrest and initiate propagation is a crack-like fashion. More recently, Needleman\(26\) considered the case of a softening, rate-dependent von Mises solid. This analysis suggests that, for solids obeying normality, the overall features of dynamic shear band growth are not unlike those found in the static case.

In this work, we investigate the effect of rate sensitivity and inertia on the development of shear bands in soils. The analysis is based on a finite element method proposed by Ortiz, Leroy and Needleman,\(14\) suitably extended to account for rate dependency. As in the static case, we show that shear banding may induce severe softening of the specimen even as the material steadily hardens. As expected, rate sensitivity tends to retard and stabilize, but not altogether eliminate, the post-peak structural softening. In addition, it is verified that the static solution is indeed recovered as the viscous limit.

The effect of inertia, by contrast, is found to be far more dramatic in frictional solids than the results of Needleman\(26\) indicate it to be for the case for solids obeying normality. Thus, under dynamic conditions, shear bands are observed to propagate discontinuously, arresting and resuming propagation repeatedly before linking up with the boundary of the specimen. The direction of the band is equally unsteady. In addition, multiple shear banding, with the development of secondary and even tertiary bands, appears to be a prevalent mechanism at sufficiently high impact velocities.

**ANALYTICAL CHARACTERIZATION OF LOCALIZATION INSTABILITIES**

In this section, we collect some basic results pertinent to the formulation of the numerical procedures to be used in the analysis. We differentiate between localization in rate-independent and rate-dependent solids. In the former, localization is associated with a change of type of the governing field equations.\(27,28\) Thus, in the static case, the equations lose ellipticity and characteristic directions arise which can carry strain discontinuities or 'shocks'. In the dynamic case, some or all of the wave speeds may become imaginary. In rate-dependent solids, by way of contrast, the principal part of the field equations is always given by the elastic Navier operator, and the equations never change type. In this case, however, a linearized instability analysis provides information on the earliest time at which a small perturbation may grow exponentially.
Localization in rate-independent solids

Consider a rate-independent solid being deformed dynamically. We seek to determine conditions under which strain discontinuities, or 'shocks' may arise as a result of a local material instability. We start by noting the appropriate jump conditions,

\[
\begin{align*}
[\ddot{u}_{ij}] &= \dot{g}_i n_j \\
\{\sigma_{ij}\} n_j + \rho V [\ddot{u}] = 0
\end{align*}
\]  

(1)

where \( u \) signifies the displacement field, \( u_{ij} \) the corresponding displacement gradients, \( \sigma_{ij} \) the components of the stress tensor and \( \rho \) the mass density. The brackets \([ \cdot ]\) denote the jump of the argument across the surface of discontinuity, regarded as an oriented surface of normal \( n_j \). The vector \( g_i \) is the polarization vector of the shock and \( V \) the relative normal velocity of the surface of discontinuity with respect to the material. The first of equations (1) is Maxwell's compatibility condition, whereas the second establishes the conservation of linear momentum across the shock.

Consider now incremental constitutive equations appropriate to rate-independent solids undergoing small deformations:

\[
\ddot{\varepsilon}_{ij} = D_{ijkl} \dot{\varepsilon}_{kl}
\]  

(2)

where \( D \) are the tangent moduli. In plastic solids, \( D \) possesses a plastic loading and an elastic unloading branch. Here we adopt Hill's linear comparison solid \(^{29}\) consisting of taking the loading branch of \( D \) on both sides of the discontinuity. For solids obeying normality, this criterion is known to provide the earliest bifurcation point.\(^{20}\)

The jump conditions (2) may be simplified by noting that the surfaces of discontinuity sought here are material surfaces for which \( V \) vanishes identically. Thus (1) is reduced to

\[
\begin{align*}
[\ddot{u}_{ij}] &= \dot{g}_i n_j \\
\{\sigma_{ij}\} n_j &= 0
\end{align*}
\]  

(3)

Remarkably, the inertia of the body is nowhere reflected in equations (2) and (3). From this we conclude that the localization conditions are not affected by inertia.

Inserting equation (2) into (3) and requiring that \( g_i \neq 0 \), one arrives at

\[
\det(A(n)) = 0
\]  

(4)

as a necessary condition for a jump to be possible along the direction \( n \). In (4),

\[
A_{ij}(n) = D_{ijkl} n_k n_l
\]  

(5)

is the acoustic tensor. These conditions are formally identical to those appropriate to static conditions\(^{27, 28}\).

It bears emphasis that (4) is a pointwise condition which can be elucidated on the basis of the state of the body at a material point. We also note that the left-hand side of (4) is the discriminant of the principal part of the static field equations. Thus, satisfaction of (4) is tantamount to a change of type of the governing equations. In the static case, the equations lose ellipticity and the incipient characteristic directions coincide with \( n \). Typically, two surfaces of discontinuity pair up to form a shear band.

For rate-independent solids undergoing adiabatic quasistatic deformations, the thickness of the band cannot be determined from the field equations, which lack a characteristic dimension. For dynamic loading of rate-dependent solids, by contrast, Needleman\(^{24}\) has shown that the elastic wave speeds coupled with the characteristic relaxation time of the material set a length scale for the deformation. It should also be noted that, as pointed out by Needleman,\(^{24}\) the governing
equations for rate-dependent solids of the viscoplastic type never loose ellipticity and, consequently, their solution is unique. In addition, owing to the ellipticity of the governing equations, defects set a length scale for the solution. These two cases, which constitute the main focus of the work reported here, provide two examples of problems with well-defined characteristic lengths in which the constitutive response is taken to be that of a local or simple continuum. Another physically motivated example is furnished by thermal softening coupled with heat conduction.\textsuperscript{20} In all of these cases, numerical solutions are free of spurious mesh size dependencies, provided that the appropriate length scales are adequately resolved by the mesh.

As an alternative possibility, other authors have sought to build a length-scale directly into the mechanical response by recourse to non-local models\textsuperscript{31–35} and to generalized continuum theories.\textsuperscript{36–39}

It should be emphasized, however, that many questions concerning the geometry of shear bands are far from being fully settled and are the subject of ongoing research. For instance, whereas the mathematical nature of the equations governing rate-dependent solids is such that defects introduce a geometrical scale, it is not known with certainty how defects influence distant portions of the shear band or how that influence evolves in time. It is also unclear the precise way in which the rate-independent and quasi-static limits, for which the influence of defects is lost, are attained. In situations close to rate-independent behaviour, and in other cases where the band thickness tends to decrease below the mesh size, the discretization does set spurious limits on the geometry of the band. These problems are unavoidable in the rate-independent case, where the bands collapse to lines of zero thickness. Under these conditions, one way of rendering the numerical results meaningful, however, is to choose the mesh size so as to match a band thickness measured experimentally or predicted by a micromechanical theory. This approach was taken Leroy and Ortiz,\textsuperscript{6} in their analysis of quasi-static experiments of Vardoulakis and Graf.\textsuperscript{40} Their solutions, as well as extensions to those solutions presented in this paper, exhibit remarkably good agreement with experiment.

\textit{Localization in rate dependent solids}

The above analysis of localization becomes trivial when applied to rate-dependent materials, owing to the fact that the principal part of the governing equations remains always elastic. As a result, the static and dynamic field equations retain their elliptic and hyperbolic character, respectively, at all times. Localization phenomena are also found in rate-dependent solids, but in the form of unstable growth of shearing modes of deformation. In this section, we show that the earliest time at which localization becomes possible can be detected by means of a linearized stability analysis of the type conducted by Anand et al.\textsuperscript{41} The same analysis suffices to determine the geometry of the earliest possible localized mode. As will become apparent in subsequent developments, this is precisely the information that is required to extend the finite element method of Ortiz, Leroy and Needleman\textsuperscript{14} to the rate-dependent range.

It should be emphasized that, as noted by Molinari and Clifton,\textsuperscript{20} a linearized stability analysis generally provides a necessary but not a sufficient condition for localization. In particular boundary value problems, the unstable mode may be stabilized by non-linear effects not taken into account in the linearized analysis. As a result, the actual growth of the unstable mode may be much delayed relative to its theoretical inception time, or in some cases arrested altogether. This prompted Molinari and Clifton\textsuperscript{20} to carry out a full non-linear stability analysis of localization. However, for the purposes at hand, primary interest lies precisely with the earliest possible time of localization, as determined from a linearized stability analysis.
We base our analysis on the following set of constitutive equations appropriate to rate-dependent plasticity:

\[
\begin{align*}
\dot{\varepsilon}_{ij} &= D_{ijkl}(\varepsilon_{kl} - \varepsilon_{kl}^0) \\
\varepsilon_{kl}^0 &= \gamma r_{ij}(\sigma, \mathbf{q}) \\
\dot{q}_a &= \dot{\gamma}h_a(\sigma, \mathbf{q}) \\
\dot{\gamma} &= \psi(\sigma, \mathbf{q})/\eta
\end{align*}
\]

where \(\sigma_{ij}\) is the stress tensor, \(\varepsilon_{ij}\) the strain tensor, \(q_a\) some suitable set of internal variables, \(\gamma\) an effective plastic strain, \(D_{ijkl}\) the elastic moduli, \(r_{ij}\) the direction of plastic flow, \(h_a\) the hardening moduli, \(\psi\) a flow potential and \(\eta\) a viscosity parameter.

We note for later reference that the rate-independent limit of equations (6) may be obtained by letting the viscosity \(\eta\) become vanishingly small. Then the boundedness of \(\dot{\gamma}\) in the fourth of equations (6) requires that

\[
\psi(\sigma, \mathbf{q}) = 0
\]

and, thus, \(\psi\) takes the significance of a yield function.

Next, we seek to ascertain under what conditions a small perturbation in the displacement field \(\delta u_i(x, t)\) can grow exponentially in time. The corresponding perturbations in stress and strain satisfy the field equations

\[
\begin{align*}
2\delta \varepsilon_{ij} &= \delta u_{i,j} + \delta u_{j,i} \\
\delta \sigma_{ij} &= 0
\end{align*}
\]

A linearization of constitutive equations (6) yields

\[
\begin{align*}
\delta \dot{\varepsilon}_{ij} &= D_{ijkl}\left\{ \delta \varepsilon_{kl} - \delta \gamma r_{ij}^0 - \dot{\gamma}^0 \left[ \frac{\partial r_{kl}}{\partial \sigma_{mm}} \right]_0 \delta \sigma_{mn} + \frac{\partial r_{kl}}{\partial q_a} \delta q_a \right\} \\
\delta \dot{q}_a &= \delta \gamma h_a^0 + \dot{\gamma}^0 \left[ \frac{\partial f_x}{\partial \sigma_{ij}} \delta \sigma_{ij} + \frac{\partial f_x}{\partial q_a} \delta q_a \right] \\
\delta \dot{\gamma} &= \frac{1}{\eta} \left[ \frac{\partial \phi}{\partial \sigma_{ij}} \right]_0 \delta \sigma_{ij} + \frac{\partial \phi}{\partial q_a} \delta q_a
\end{align*}
\]

where the superscript '0' is used to denote quantities evaluated at the reference state. Next consider perturbations undergoing exponential growth or decay, i.e.

\[
\delta u_i(x, t) = \delta \dot{u}_i(x) \exp(\lambda t)
\]

where the parameter \(\lambda\) determines the rate of growth, if \(\lambda > 0\), or the rate of decay, if \(\lambda < 0\). Inserting (10) into (9) one finds

\[
\begin{align*}
\lambda \delta \varepsilon_{ij} &= D_{ijkl}\left\{ \lambda \delta \varepsilon_{kl} - \lambda \delta \gamma r_{ij}^0 - \dot{\gamma}^0 \left[ \frac{\partial r_{kl}}{\partial \sigma_{mm}} \right]_0 \delta \sigma_{mn} + \frac{\partial r_{kl}}{\partial q_a} \delta q_a \right\} \\
\lambda \delta \dot{q}_a &= \lambda \delta \gamma h_a^0 + \dot{\gamma}^0 \left[ \frac{\partial f_x}{\partial \sigma_{ij}} \delta \sigma_{ij} + \frac{\partial f_x}{\partial q_a} \delta q_a \right] \\
\lambda \delta \dot{\gamma} &= \frac{1}{\eta} \left[ \frac{\partial \phi}{\partial \sigma_{ij}} \right]_0 \delta \sigma_{ij} + \frac{\partial \phi}{\partial q_a} \delta q_a
\end{align*}
\]
Further reduction results in a relation
\[
\delta \sigma_{ij} = L_{ijkl}(\lambda) \delta \sigma_{kl}
\]
for some moduli $L_{ijkl}$. Now we further specialize the perturbation to a form appropriate to a localization instability,
\[
\delta \dot{u}_i(x) = U(x_a)m_i
\]
\[
x_a = (x_i - x_i^\ast)n_i
\]
where $x_i^\ast$ is some reference point on the band, $n_i$ is the unit normal to the band, $m_i$ a unit polarization vector, $x_a$ the co-ordinate normal to the band, and the function $U(x_a)$ defines the normal profile of the perturbation. Substituting (13) into the compatibility equation (8a), and combining the result with the linearized constitutive relations (12) and the equation of equilibrium (8b), one finds
\[
U''L_{ijkl}(\lambda)n_in_km_k = 0
\]
(14)
From this equation one concludes that, for a non-trivial solution to exist, the normal $n_i$ must be chosen so as to satisfy the condition
\[
\det(L_{ijkl}(\lambda)n_in_k) = 0
\]
(15)
If, for a given reference state, a direction $n_i$ can be found for which the corresponding $\lambda$ following from (15) is positive, we conclude that localization in the form of a band of orientation $n_i$ is possible. The value of $\lambda$ then gives the rate of growth of the perturbation and the polarization vector $m_i$ follows as the null eigenvector of the pseudo-acoustic tensor $L_{ijkl}(\lambda)n_in_k$.

Of primary interest to the applications sought here is the limiting case of $\lambda \rightarrow 0$. Thus, we wish to ascertain under what conditions a perturbation can grow at an arbitrarily small rate. From the asymptotic properties of viscoplasticity in the inviscid limit, one would expect that slowly varying deformations be governed by the rate independent limit of the constitutive relations. That this is indeed so may be demonstrated as follows. Divide through by $\lambda$ in the first and second of equations (11) to find
\[
\delta \sigma_{ij} = D_{ijkl}(\delta \sigma_{kl} - \delta \dot{r}_{kl})
\]
\[
\delta q_a = \delta \dot{h}_a^0 + \frac{\dot{q}}{\lambda} \left[ \left( \frac{\partial F_\sigma}{\partial \sigma_{ij}} \right)_o \delta \sigma_{ij} + \left( \frac{\partial F_\sigma}{\partial q_\beta} \right)_o \delta q_\beta \right]
\]
(16)
Evidently, for these equations to remain bounded for arbitrarily small $\lambda$ one must have $\dot{q}^0$ arbitrarily close to $0^\ast$. From (6), it follows that, under these conditions, the reference state must be arbitrarily close to the yield surface (7). Thus, we conclude that arbitrarily slow perturbations in a rate-dependent solid can only grow from quasistatic solutions. Taking $\dot{q}^0 \rightarrow 0^\ast$, equations (11) reduce to
\[
\delta \sigma_{ij} = D_{ijkl}(\delta \sigma_{kl} - \delta \dot{r}_{kl})
\]
\[
\delta q_a = \delta \dot{h}_a^0
\]
\[
\delta \dot{q}^0 = \frac{1}{\lambda \eta} \left[ \left( \frac{\partial \phi}{\partial \sigma_{ij}} \right)_o \delta \sigma_{ij} + \left( \frac{\partial \phi}{\partial q_\beta} \right)_o \delta q_\beta \right]
\]
(17)
For this case, the moduli $L_{ijkl}$ are explicitly given by

$$L_{ijkl}(\lambda) = D_{ijkl} - \frac{(D_{ijklr}^{q})((\partial \phi / \partial \sigma_{pq})^{q} D_{pqkl})}{\lambda \eta + (\partial \phi / \partial \sigma_{pq})^{q} D_{pqkl} - (\partial \phi / \partial q_{r})^{q} h_{r}^{k}}$$

(18)

Finally, effecting the limit of $\lambda \to 0^+$ one finds

$$\lim_{\lambda \to 0^+} L_{ijkl}(\lambda) = D_{ijkl} - \frac{(D_{ijklr}^{q})((\partial \phi / \partial \sigma_{pq})^{q} D_{pqkl})}{(\partial \phi / \partial \sigma_{pq})^{q} D_{pqkl} - (\partial \phi / \partial q_{r})^{q} h_{r}^{k}} \equiv D_{ijkl}^{\eta}$$

(19)

where $D_{ijkl}^{\eta}$ are the rate-independent elastoplastic moduli. Note that the rate-independent moduli also follow from (17) by taking the inviscid limit $\eta \to 0^+$.

Thus, in the limit $\lambda \to 0$ the localization condition (15) reduces to its rate-independent counterpart (4). The boundary between stable and unstable behaviour precisely corresponds to conditions for which perturbations can grow at a vanishing rate. Hence, the rate-independent limit does indeed determine the earliest possible time at which localization may take place.

Further insight into the phenomenology of localization in rate-dependent solids may be gained by means of the following example. Consider a one-dimensional viscoplastic solid obeying the linear overstress model

$$\dot{\sigma} = E(\dot{\varepsilon} - \dot{\varepsilon}^{p})$$

$$\dot{\varepsilon}^{p} = \frac{\sigma - \sigma_{0}}{\eta}$$

$$\dot{\sigma}_{0} = h\dot{\varepsilon}^{p}$$

(20)

where $h$ is the plastic modulus. A minor computation leads to the expression

$$L = \frac{1}{1 + \frac{1}{E} \frac{1}{\lambda \eta + h}}$$

(21)

for the sole non-zero component of $L_{ijkl}$. Under these simplified conditions, the localization condition (15) reduces to $L = 0$, which necessitates

$$\lambda \eta + h = 0$$

(22)

In the inviscid limit, $\eta = 0$, this reduces to the well-known result that localization occurs at the peak stress. For non-zero $\eta$, equation (22) may be interpreted in several ways. For a given rate $\lambda$, one computes the plastic modulus required to sustain growth as $h = -\lambda \eta$. As may be seen, $h$ needs to be become increasingly negative as the rate of growth and the viscosity are increased. For a given plastic modulus $h$, equation (22) yields the rate of growth of the perturbation, $\lambda = -h/\eta$. As may be seen, growth requires the plastic modulus to be negative.

**FINITE ELEMENT FORMULATION**

In problems involving strong strain localization, the performance of conventional isoparametric quadrilaterals can be substantially improved by adding structure conferring the element the ability to develop strain discontinuities, or 'shocks', of arbitrary orientation and polarization. This approach has been proposed by Ortiz, Leroy and Needleman and by Nacar, Needleman and Ortiz. Applications of the method to frictional materials have been given by Leroy and Ortiz.
The approach applies quite generally to static and dynamic analyses of two- and three-dimensional solids obeying rate-dependent or rate-independent behaviour and undergoing small or large deformations. For completeness, some highlights of the method are given below, with particular emphasis on matters of implementation within the framework of explicit dynamic computations.

Consider an incremental finite-element analysis of an inelastic body expected to undergo strain localization. As discussed in the preceding section, by virtue of the very nature of localization instabilities the onset of localization can be detected pointwise at, say, the reduced quadrature points within each element. The geometry of the incipient localized modes is also found readily in the same manner. For rate-independent solids, localization is detected by monitoring the eigenvalues of the acoustic tensor. For rate-dependent solids of the viscoplastic type, a similar analysis based on the tangent moduli of the limiting rate-independent material yields the earliest time at which localization becomes possible.

Once localization is detected within an element, incompatible shape functions possessing appropriate strain discontinuities are added to the element interpolation, which thus becomes

\[ u_i(x) = \sum_a u_{ia} N_a(x) + \sum_a g_a M_{ia}(x) \]  

(23)

where \( u_i \) are the displacement fields within the element, \( n_{ia} \) and \( N_a \) the corresponding nodal displacements and shape functions, \( M_{ia} \) the localized modes and \( g_a \) the corresponding amplitudes. The sum on \( a \) extends over those reduced quadrature points within the element where localization is in progress.

The idea of adding structure to an element in order to ameliorate deficient properties has a long standing in the finite element literature. Following the pioneering work of Wilson et al., who added two bending modes to an isoparametric quadrilateral to improve its bending properties, numerous ‘incompatible’ elements have been proposed in a variety of fields of application.

The localized modes \( M_{ia} \) are chosen to be of the form

\[ M_{ia}(x) = \chi_a(x) m_{ia} n_{ja} (x_j - \xi_{ja}) \]  

(no sum on \( a \))  

(24)

where \( \xi_{ja} \) are the co-ordinates of the reduced quadrature point under consideration, \( n_{ja} \) the normal to the discontinuity surface and \( m_{ia} \) the associated polarization vector. These latter two vectors are found readily by recourse to a pointwise localization analysis. The discontinuous function \( \chi_a(x) \) is defined as

\[ \chi_a(x) = \begin{cases} \lambda_a, & \text{if } (x_i - \xi_{ia}) n_{ia} > 0 \\ -(1 - \lambda_a), & \text{if } (x_i - \xi_{ia}) n_{ia} < 0 \end{cases} \]  

(25)

for an as yet undetermined parameter \( \lambda_a \).

The motivation for this choice of interpolation stems from the rate-independent case, for which shear bands are bounded by shocks of the form expressed in (1). By virtue of the jump built into \( \chi_a \), the displacement gradients at any point on the surface \( n_{ia}(x_i - \xi_{ia}) = 0 \) exhibit the jump

\[ [u_{i,j}(x)] = g_a m_{ia} n_{ja} \]  

(26)

which is precisely of the form (1), with \( g_a = g_a m_{ia} \). It should be emphasized that the additional degrees of freedom \( g_a \) are internal to the element and can be eliminated by static condensation.

Several variations of this procedure are possible. Nacar, Needleman and Ortiz, for instance, employed an a priori material instability analysis to determine the proper orientation of the localized modes. In some applications, it has been found that the performance of the element is improved by activating the added modes immediately following yielding. In this case, the
Localization directions are determined based on the minima, rather than the zeros, of the determinant of the acoustic tensor. Finally, in applications involving frictional materials, a periodic re-evaluation of the localization directions is required to obtain sharp results.6

The discretized equations of equilibrium of the element may be formulated as follows. For purposes of illustration, we consider the case of a non-linear elastic solid. Extensions to plasticity are straightforward. Let

\[ J_k = \int_{\Omega^e} W(\varepsilon) d\Omega - \int_{\Gamma^e} b_i u_i d\Gamma - \int_{\Gamma^e_t} t_i u_i d\Gamma \quad (27) \]

be the potential energy of the element. Here, \( \Omega^e \) denotes the domain of the element, \( \Gamma^e_t \) its traction boundary, \( W \) the strain energy density of the body, \( b_i \) the body forces and \( t_i \) the boundary tractions. The strain field \( \varepsilon \) is computed from (23) to be of the form

\[ \varepsilon(x) = B_1(x) u + B_2(x) g \quad (28) \]

where \( u \) and \( g \) are the arrays of nodal displacements and localized mode amplitudes, respectively, and \( B_1 \) and \( B_2 \) play the role of discrete symmetric gradient operators. In cases where locking due to the near-incompressibility is a concern, the compatible \( B_1 \) matrix may be simply replaced by a suitably modified operator, say \( B_{1'} \), as proposed by Hughes.53

Next, we seek to minimize \( J_k \) with respect to all the kinematic degrees of freedom of the element, namely, \( u \) and \( g \). Rendering \( J_k \) stationary with respect to \( u \), we find

\[ -F_1 \equiv \int_{\Omega^e} B_1^T \sigma d\Omega - \int_{\Gamma^e} N^T b d\Gamma - \int_{\Gamma^e_t} N^T t d\Gamma = \int_{\Omega^e} B_1^T \sigma d\Omega - f_1 = 0 \quad (29) \]

where \( \sigma = \partial W/\partial \varepsilon \). On the other hand, establishing the stationarity of \( J_k \) with respect to \( g \) one finds

\[ -F_2 \equiv \int_{\Omega^e} B_2^T \sigma d\Omega = 0 \quad (30) \]

Equations (29) and (31) are the sought equations of equilibrium of the element. It is interesting to note that the forces \( F_2 \) conjugate to the localized modes arise as a set of self-equilibrated forces internal to the element. In the context of explicit integration, the static condensation of \( g \) may be effected from the linearized equation of equilibrium

\[ \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix} \begin{bmatrix} \Delta u \\ \Delta g \end{bmatrix} = \begin{bmatrix} f_1 - F_1 \\ -F_2 \end{bmatrix} \quad (31) \]

In introducing the added localized modes, care must be exercised to ensure a smooth transition in the computed out-of-balance forces. It follows from (31) that the nodal out-of-balance forces remain continuous though the transition if and only if the condition

\[ \int_{\Omega^e} B_1^T \sigma d\Omega = 0 \quad (32) \]

is satisfied at localization. This requirement determines uniquely the parameters \( \lambda_x \) in the definition (25) of the embedded localized modes. If the stresses at localization are uniform over the element, equation (32) reduces to

\[ \int_{\Omega^e} B_1^T \sigma d\Omega = 0 \quad (33) \]

This has been shown by Taylor et al.44 to be a sufficient condition for the satisfaction of the patch
test of Irons\textsuperscript{45} by incompatible elements, such as those resulting here from the addition of the embedded localized modes.

We illustrate the effect of the added modes on the performance of the element with the aid of a simple one-element example (Figure 1). The problem concerns a Mises solid with parabolic softening subjected to plane strain tension. The parameters employed in the calculations are a Young's modulus $E = 3 \times 10^6$, a Poisson's ratio $v = 0.3$, a yield stress $\sigma_0 = 60$, an initial hardening modulus $h_0 = 10^3$, a curvature of the stress-strain curve $\kappa = -10^6$ and a cut-off stress $\sigma_c = 2$. An initial inhomogeneity in the form of a 10 per cent reduction of the yield stress was introduced at the first quadrature point (Figure 1). Hughes' B-method was adopted to prevent locking due to near-incompressibility under fully developed plastic flow.

Figure 2 depicts the homogeneous response of the element, as well as those computed from the isoparametric and the enhanced elements. Localization was monitored at the centroid and, as expected, found to occur at $\pm 45^\circ$ to the direction of loading. As may be seen, the enhanced element results in stronger overall softening of the element (Figure 2). The differences between the

![Figure 1. One-element example. Inlaid are localization directions](image)

![Figure 2. Load-displacement curves for the one-element problem](image)
two methods are clearly manifest in the computed histories of the strain component $\varepsilon_{22}$ (Figure 3). Whereas the strains remain uniform throughout the element in the isoparametric results, following localization the enhanced element predicts a deformation pattern increasingly localized to the upper portion of the element. The histories of the amplitudes of the incompatible modes are shown in Figure 4. As expected, the localized mode activity exhibits a sharp increase immediately following the inception of localization. Finally, Figure 5 illustrates the effect of rate dependency on the performance of the method. The constitutive equations were extended into the rate-dependent range with the aid of a linear overstress model of the form (6). As may be seen, the main features of the response of the element remain unaltered as rate dependence is introduced, but with an increasing delay in the unstable branch of the force-displacement relation (Figure 5).

![Figure 3](image1.png)

Figure 3. Histories of $\varepsilon_{22}$ at integration points for the one-element example

![Figure 4](image2.png)

Figure 4. Histories of the amplitudes of the added localized modes for the one-element example
Figure 5. Load-displacement curves for one-element example showing the effect of rate sensitivity

**Extension to dynamic conditions**

In cases where inertia is not negligible, the equilibrium equations needs to be replaced by the equations of motion

\[
\int_{\Omega^e} \mathbf{B}^T \sigma d\Omega + \mathbf{M} \ddot{\mathbf{u}} = \int_{\Omega^e} \mathbf{N}^T \mathbf{b} d\Omega + \int_{\Gamma^e} \mathbf{N}^T \mathbf{t} d\Gamma
\]

\[
\int_{\Omega^e} \mathbf{B}^T \tau d\Omega = 0
\]

(34)

where \( \mathbf{u} \) is the nodal acceleration vector and \( \mathbf{M} \) is some suitable mass matrix. Note that no mass is associated with the localized modes. Thus, the static condensation procedure described above carries over to the dynamic case.

For the class of impact problems considered here, explicit methods of integration are advantageous. In the calculations that follow, we have adopted Newmark's method with \( \beta = 0 \) and \( \gamma = 1/2 \). The conventional implementation of the method needs to be modified to deal with the added localized modes. A flow-chart of the resulting algorithm is given in Table I. For rate-dependent solids, the static condensation step poses some difficulties, owing to the fact that no natural stiffness exists other than the elastic. In this case, the static condensation procedure was implemented in a way essentially equivalent to computing the arrays \( \mathbf{K}_{22} \) and \( \mathbf{K}_{23} \) based on the rate tangent modulus of Reference 46. We emphasize that all operations pertaining to the handling of the added modes are effected at the element level.

**CONSTITUTIVE MODEL**

Pressure-sensitive yield and non-normality are two key constitutive features characteristic of frictional materials such as soils, rocks and concrete. The strongly dilatant, non-associated character of the plastic flow suffices by itself to trigger localization instabilities, even when the material is in a state of hardening.\(^9,10\) Furthermore, the shear strength of these materials is generally debilitated by the application of hydrostatic tension. The marked influence of pressure


sensitivity on the post-localization response of soil samples has been documented experimentally by Hettler and Vardoulakis, and is borne out by numerical simulations.

In this work, we adopt a variant of Drucker and Prager's model which incorporates these essential features in a simple form particularly well suited to computation. For sufficiently high confining pressure, the yield function and flow potential are assumed to take the form

\[ F(\sigma) = Q + \alpha(P - P_0) \leq 0 \]  

\[ G(\sigma) = Q + \beta P \]  

where \( P_0, \alpha \) and \( \beta \) are material constants, \( \sigma \) is the stress tensor and

\[ P = \sigma_{kk}/3 \]

\[ Q = \sqrt{[(3/2)S_{ij}S_{ij}], \quad S_{ij} = \sigma_{ij} - P \delta_{ij}} \]  

are the hydrostatic pressure, the effective shear stress and the stress deviator, respectively. Thus, states of stress such that \( F < 0 \) are assumed to result in an elastic response, and the direction of plastic flow is taken to coincide with the gradient \( \partial G/\partial \sigma_{ij} \). Note that the plastic flow is rendered non-associated for choices of the parameters such that \( \beta \neq \alpha \).

The constants of the model may be placed in correspondence with the Mohr-Coulomb parameters, \( c, \phi, \psi \) by requiring that the two models coincide for the triaxial test. Here \( c \) is the cohesion of the material, \( \phi \) the angle of internal friction and \( \psi \) the dilatancy angle. This matching results in the relations

\[ \alpha = \frac{6 \sin \phi}{3 - \sin \phi}, \quad c = P_0 \tan \phi, \quad \beta = \frac{6 \sin \psi}{3 - \sin \psi} \]  

The material is assumed to undergo hardening in the form of a monotonic increase in the angle of friction from an initial value \( \phi_i \) up to a saturation level \( \phi_m \) attained at a critical effective plastic strain \( \gamma_e \). The precise functional form of the transition is chosen to be

\[ \sin \phi = \sin \phi_i + \frac{k \sqrt{\gamma_e}}{\gamma + \gamma_e}, \quad k = 2(\sin \phi_m - \sin \phi_i) \sqrt{\gamma_e} \]  

Table I. Explicit integration

(i) Initialization: \( n = 0, a_0 = M^{-1}(f_1^0 - F_i^0) \)

(ii) Predictor:

\[ u_{n+1} = u_n + \Delta t v_n + (1/2)\Delta t^2 a_n \]

(iii) Static condensation:

\[ \Delta g = -K_{zz}^{-1} (K_{1z} a + F_1) \]

\[ g_{n+1} = g_n + \Delta g \]

(iv) State update: \( u_{e+1}, a_{e+1}, q_{e+1}, \) bifurcation check

(v) Out-of-balance forces: \( r_1^{e+1} = f_1^{e+1} - F_1^{e+1} \)

(vi) Accelerations:

\[ a_{e+1} = M^{-1} r_1^{e+1} \]

(vii) Velocities:

\[ v_{e+1} = v_e + (1/2)\Delta t (a_e + a_{e+1}) \]

(viii) Stability check

(ix) \( n = n + 1 \), go to (ii)
Here, the effective plastic strain $\gamma$ is defined as

$$\gamma = \int_0^t \sqrt{[\frac{2}{3} \dot{\varepsilon}_T^p \dot{\varepsilon}_T^p]} \, dt$$

(39)

In cases where one wishes to investigate the effect of material softening, we extend the hardening law up to a transition effective plastic strain $\gamma^* > \gamma_0$. Beyond the transition strain we adopt the law

$$\sin \phi = \sin \phi_f + K \exp\left(-\frac{\gamma}{\gamma_0}\right), \quad \gamma > \gamma^*$$

(40)

where $\phi_f < \phi_m$ is the lower limit for the friction angle and the constants $K$ and $\gamma_0$ are chosen so as to obtain a continuously differentiable transition between (38) and (40).

A difficulty associated with the yield function (35a) is that it exhibits a corner at the point $P = P_0$ on the hydrostatic axis, corresponding to failure by cavitation. From a numerical standpoint, the presence of the corner may render the state update procedure ill-posed or even undefined when the confining pressure is small. To overcome these problems, we round-off the corner by fitting spherical caps of the form,

$$F(\sigma) = H_f \{ \sqrt{[Q^2 + (P - (1 - \delta) P_0)^2]} - R \}, \quad \alpha Q \geq P - (1 - \delta) P_0$$

$$G(\sigma) = H_g \{ \sqrt{[Q^2 + (P - (1 - \delta) P_0)^2]} \}, \quad \beta Q \geq P - (1 - \delta) P_0$$

(41)

The composite yield function and flow potential are shown in Figure 6. The parameters $H_f$, $R$ and $H_g$ are chosen so as to ensure a continuously differentiable transition into the caps.

The idea of rounding-off the corner by means of a cap was already suggested in the original paper of Drucker and Prager.\textsuperscript{48} As regards the flow potential, the effect of the tensile cap mimics a regain of dilatancy under cavitation conditions. Interestingly enough, the experimental observations of Hettler and Vardoulakis\textsuperscript{47} and Desrues\textsuperscript{2} exhibit precisely that feature.

**NUMERICAL RESULTS**

By way of background and as a basis for comparison, we start by giving a detailed analysis of the plane strain uniaxial compression test in a rate-independent solid under quasistatic loading. The analysis extends that given by Leroy and Ortiz.\textsuperscript{6} The extension is made possible by the adoption of
a tension cap, which enables the computations to be continued beyond the onset of cavitation in the band. Subsequent calculations are intended to elucidate the effect of rate dependency and inertia on the process of localization.

**Rate-independent results**

Our first example concerns the plane strain uniaxial compression test on a rectangular specimen. The material is assumed to be rate independent and to undergo quasistatic small deformations. For the purpose of these calculations, the hardening law (38) with no subsequent softening is adopted. The material constants are chosen as \( E = 2 \times 10^8 \), \( \nu = 0.25 \), \( \phi_l = 10^\circ \), \( \phi_m = 25^\circ \), \( \gamma_c = 0.5 \) per cent, \( \delta = 0.1 \) and \( \psi = 0^\circ \). This choice of dilatancy angle renders the plastic flow incompressible and strongly non-associated. Hughes’ \( \mathbf{B} \)-method\(^{42} \) is adopted to prevent locking due to near-incompressibility under fully developed plastic flow. An imperfection in the form of a softer element with \( \phi^m = 17^\circ \) is introduced at the centre of the specimen. The twofold symmetry of the problem permits restricting the analysis to one quarter of the specimen. The calculations are based on the mesh shown in Figure 7.

Figure 8 shows the force–displacement curves computed for three values of the confining pressure, \( P_0 = 10^4 \), \( 2 \times 10^4 \) and \( 3 \times 10^4 \). Also shown for comparison are the corresponding uniform solutions. It is observed that, initially, the uniform and perturbed solutions ostensibly coincide. As the prescribed displacements are increased, a shear band propagates from the imperfection (Figures 10 and 11). As soon as the shear band links up with the surface of the specimen, the load–displacement curve suffers an abrupt drop from the level corresponding to the uniform solution (Figure 8). Eventually, the load-bearing capacity of the specimen stabilizes at a lower plateau. These results are in close qualitative and quantitative agreement with the experimental observations of Vardoulakis.\(^{47} \) It is interesting to note that, owing to the development of the shear band, a strongly softening response of the specimen is obtained, even as the material remains hardening at all times. This illustrates the difficulties associated with a direct interpretation of test results beyond the onset of localization.

![Figure 7. Localized elements for static rate independent plane strain uniaxial compression test. Shown is one quarter of the specimen](image)
Figure 8. Load–displacement curve for static, rate-independent plane strain uniaxial compression test

Figure 9. Load–displacement curve for static, rate-independent plane strain uniaxial compression test scaled by confining pressure

Figure 9 shows the three load–displacement curves with the loads scaled by the corresponding confining pressure. As may be seen, the lower plateau of all three curves are at roughly a common value of 1. This strongly suggests that the force–displacement curve for the plane strain uniaxial compression test stabilizes at the level of the confining pressure. Interestingly, an analysis of the problem based on an infinite band in an unbounded solid grossly overestimates the level of the lower plateau. From these observations, it appears that, for the problem at hand, size effects and the boundary conditions play an important role in the post-peak range and need to be taken into account in the analysis.

Figures 10 and 11 show in detail the fully developed shear band. The results correspond to the intermediate confining pressure $P_0 = 2 \times 10^5$. As may be seen, the enhanced element sharpens
the strain gradients to the full resolution of the mesh. By contrast, the numerical tests of Leroy and Ortiz\textsuperscript{6} demonstrate that isoparametric elements inhibit the formation of the shear band altogether, irrespective of whether steps are taken to eliminated locking due to near-incompressibility.
Effect of inertia

To ascertain the effect of inertia on localization, we repeat the above analysis under dynamic conditions. Velocities are prescribed on the upper boundary, according to a ramp variation

\[ v(t) = \begin{cases} \frac{t}{t_0} & t \in [0, t_0] \\ V_0 & t > t_0 \end{cases} \]

where the rise time \( t_0 \) is chosen to be equal to twice the initial time step based on an elastic Courant number of 1. Two impact velocities \( V_0 = 0.1, 0.3 \) are considered, corresponding to a homogeneous strain rate of approximately 0.5 and 1.5, respectively. In terms of the elastic longitudinal wave speed \( c_p \), the two velocities correspond to values of \( c_p/1200 \) and \( c_p/3500 \), respectively.

The time step used in the calculations is initially set to give an elastic Courant number of 1. Subsequently, the accuracy of the time-stepping algorithm is checked by monitoring the balance of energy in the solid, as suggested by Belytschko. The time step is automatically adjusted so that the energy check is satisfied to within a fraction of a per cent.

The results corresponding to the lower impact velocity are shown in Figure 12–15. The overall features of the solution are qualitatively similar to those of the static results. Thus, Figure 12 compares the force–displacement curves for the static and dynamic problems. As may be seen, at the impact velocity under consideration the effect of inertia is rather minor. In essence, the static solution is recovered with a slight delay in the formation of the band. The contour levels of effective plastic strain (Figure 15) also show rather minor variations with respect to the static solution, in the form a slight bending of the band in the vicinity of the free surface.

Figures 16–19 show the results of the computations for the higher impact velocity. The effect of inertia is now quite clearly apparent. For instance, a substantial delay is observed in the appearance of the unstable branch of the force–displacement curve. In addition, it was observed that the band, as defined by the set of elements beyond the localization threshold, propagated discontinuously, arresting and resuming propagation several times before reaching the boundary. These observations are in keeping with the analysis of Freund et al. An inspection of the localization directions (Figure 19) also reveals some degree of zig-zagging in the orientation of the band.
Figure 13. Localized elements for dynamic rate-independent plane strain uniaxial compression test, low impact velocity. Shown is one quarter of the specimen

Figure 14. Deformed mesh for dynamic-rate independent plane strain uniaxial compression test, low impact velocity

band. This may be due to the fact that the various portions of the band grow under quite different stress conditions, owing to the repeated passages of the stress waves. Another salient departure from the static solution may be appreciated in Figures 18 and 19 in the form of a secondary band linking up with the main band.

To gain an understanding of the effect of the mesh on the various features of the solution, the above test is repeated for a finer grid (Figures 20–23). The effect of mesh refinement on the load–displacement curve (Figure 20) is found to amount to a slight reduction in the localization
Figure 15. Contours of effective plastic strain for dynamic rate-independent plane strain uniaxial compression test, low impact velocity.

Figure 16. Load–displacement curves for dynamic rate-independent plane strain uniaxial compression test, high impact velocity.

delay and a somewhat steeper unstable branch. The contours of effective plastic strain and localization directions are in rather close agreement with those computed from the coarser mesh, except for the appearance of tertiary band emanating from the secondary one. Extrapolating these results, it is tempting to surmise that, whereas the overall response of the specimen may be quite insensitive to the choice of grid, multiple shear banding may be inhibited in the coarser meshes. Conversely, mesh refinement seems to promote the development of increasingly intricate shear band patterns.
Static rate-dependent examples

Next, we focus on the effect of rate dependency. The model used in the preceding calculations can be extended simply into the rate-dependent range by means of a linear overstress model of the type (6). Figure 24 illustrates the effect of an increasing rate dependency on the load–displacement response of the plane strain uniaxial compression test. Aside from the non-vanishing viscosity, the remaining aspects of the analysis are as in previous examples. It is observed that a high rate dependency introduces a considerable overshooting of the peak load and an overall delay in the
progress of localization. The overshooting in the force level carries over into the lower plateau of the unstable branch. Conversely, as the viscosity of the material is allowed to become vanishingly small, the rate-independent results are recovered.

An additional effect of rate dependence is a significant broadening of the band (Figures 25–28). This situation underscores the need for numerical schemes capable of producing sharp shear bands to the full resolution of the mesh. Indeed, the use of an overly diffusive method of analysis greatly compounds the interpretation of results, as it is frequently unclear whether the broadening of the band is a real feature of the solution or a numerical effect.
Figure 21. Localized elements for dynamic rate-independent plane strain uniaxial compression test, high impact velocity. Shown is one quarter of the specimen. Refined mesh.

Figure 22. Deformed mesh for dynamic rate-independent plane strain uniaxial compression test, high impact velocity. Refined mesh.

Dynamic rate-dependent examples

Finally, we repeat the above analysis with both inertia and rate dependence. The impact velocity is set to \( V_0 = 0.3 \), corresponding to the highest value considered in the dynamic, rate-independent calculations. The viscosity is chosen so that \( \dot{\eta} / \dot{\gamma}_0 = 1.5 \times 10^{-2} \), where \( \dot{\gamma} \) is the macroscopic strain rate. This value is in the range considered in the static rate-dependent analysis.

Figure 29 collects the load–displacement curves for the uniform, dynamic rate-dependent, static rate-dependent and dynamic rate-independent cases. Interestingly enough, it is found that the
dynamic rate-dependent and rate-independent solutions run very close up to localization, the former overshooting the latter by a slight amount. By contrast, following localization the two solutions drift considerably apart from each other, the dynamic rate-dependent solution showing the combined delaying effects of inertia and rate dependency.

As regards the contours of effective plastic strain (Figure 30), rate dependency appears to cause a gradual broadening of the band away from the defect. The extent of broadening reaches a maximum at the free surface of the specimen. The broadening of the band is also apparent in the
Figure 25. Contours of effective plastic strain for static rate-dependent plane strain uniaxial compression test, low viscosity.

Figure 26. Localized elements for static rate-dependent plane strain uniaxial compression test, low viscosity. Shown is one quarter of the specimen.

deformed mesh (Figure 31). The localized elements (Figure 32), show the same fragmentation pattern as in the rate-independent case. However, in the rate-dependent solution the secondary and tertiary bands develop with a certain delay relative to their rate-independent counterparts. In addition, the orientations of the secondary and tertiary bands differ somewhat in both solutions.
Figure 27. Contours of effective plastic strain for static rate-dependent plane strain uniaxial compression test, high viscosity

Figure 28. Localized elements for static rate-dependent plane strain uniaxial compression test, high viscosity. Shown is one quarter of the specimen

A three-dimensional example

Our last example is intended to illustrate the effect of specimen geometry and boundary conditions on localization. The analysis concerns a triaxial test of a sample obeying the same constitutive laws considered in the above two-dimensional examples. For the class of granular
Figure 29. Load-displacement curves for dynamic rate-dependent plane strain uniaxial compression test, high impact velocity. Refined mesh.

Figure 30. Contours of effective plastic strain for dynamic rate-dependent plane strain uniaxial compression test, high impact velocity. Refined mesh.

materials considered here, Vardoulakis$^1$ and Desrues$^2$ observed that triaxial samples deform predominantly in a barrelling mode, with no evidence of localization up to very large imposed deformations. This behaviour is in sharp contrast to that of plane strain specimens, where localization occurs readily after the early stages of deformation.

A similar situation is found in the tensile test in metals, where failure is often not by shear banding but rather by cup–cone rupture. An analysis of Tvergaard and Needleman$^{34}$ reveals that the conditions of high triaxiality prevalent at the core of the specimen inhibit shear banding in
favour of ductile rupture. Only when the resulting crack propagates close enough to the free surface, where it meets with increasingly plane strain conditions, do shear lips start to form.

To facilitate comparisons with the two-dimensional results, the aspect ratios of the two specimens are taken to be roughly coincident. A material imperfection is introduced at the bottom
of the specimen (Figure 33), which renders the analysis fully three dimensional. A velocity $V_0 = 0.1$ is applied at the top of the specimen, so as to render the sample in compression. A small viscosity of $\eta = 10^3$ was used. For this choice of parameters, the response of the specimen approaches quasistatic, rate-independent conditions. Details pertaining to the implementation of the finite element method in three dimensions may be found in the paper of Leroy and Ortiz.\textsuperscript{6}

A first test simulates frictionless supports and employs a small confining pressure $P_0 = 10^5$, (Figure 34, curve A). Under these conditions, no indication of localization is found, either in the form of structural softening or of local element bifurcations. The specimen remains ostensibly

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig33}
\caption{Mesh employed in triaxial test example}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig34}
\caption{Load-displacement curves for triaxial test example}
\end{figure}
cylindrical at all times, despite the presence of the imperfection. These results are in keeping with the analysis of Rudnicki and Rice, who noted that, under triaxial conditions, softening in the constitutive law is a requirement for localization.

Curve C in Figure 34 corresponds to conditions of perfect stick at the supports. Friction between the specimen and the platens has long been recognized to be a coadjuvant factor in the initiation of localization instabilities. Again, no recognizable signs of structural softening are apparent from the solution, neither do any elements undergo bifurcation. In contrast to the frictionless case, however, the specimen deforms in a bulging mode (Figure 35), sometimes referred to as diffused localization. Note, however, that this mode of deformation is a direct consequence of the boundary conditions and not the result of an instability, material or geometrical.

Finally, we investigate the effect of softening in the constitutive law. To this end, we choose a lower friction angle $\phi_f = 15^\circ$ and transition strain $\gamma_* = 0.9$ percent in equation (40). Despite the introduction of softening, the results obtained are qualitatively identical to those for the hardening material. Thus, all elements remain under the bifurcation threshold, no geometrical softening arises, and the modes of deformation, cylindrical and bulging are as those obtained under conditions of hardening (Figure 35).

In conclusion, for the range of parameters investigated here, our results are in keeping with the experimental observations of Vardoulakis and Desrues, namely that localization instabilities do not arise in the triaxial test under small strain conditions. This points to the triaxial test as being better suited for material identification than the plane strain uniaxial compression test, where localization occurs readily. In particular, the occurrence of unstable behaviour in the triaxial test at small strains can be safely construed as an indication of material softening.

REFERENCES

LOCALIZATION IN FRICTIONAL SOLIDS


Borehole inverse pr that does constituti
This ap which inc are model closure di spaced of

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