FINITE ELEMENT ANALYSIS OF STRAIN LOCALIZATION IN FRICTIONAL MATERIALS

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SUMMARY

Numerical examples are given which illustrate the poor performance of conventional finite elements in problems involving strain localization in frictional materials. In one of the cases investigated, that of granular media subjected to plane strain biaxial loading, isoparametric elements are seen to inhibit localization altogether. With these examples by way of motivation, the performance of a recently proposed finite element method in the context of strain localization in frictional materials is assessed, with particular emphasis on three-dimensional problems. In passing, some issues pertaining to the post-bifurcation response of biaxial specimens are examined. In particular, the numerical simulations suggest that the observed softening is a geometrical effect not attributable to constitutive behaviour.

1. INTRODUCTION

The development of bands of intense inelastic deformation is a common occurrence in frictional materials such as soils, rocks and concrete. In rocks, narrow bands of localized deformation are observed following compressive failure, both in laboratory experiments and, naturally, as earth faults. In clays, shear bands are seen to form during the triaxial test and in plane strain compression. The appearance of shear bands in soil samples is frequently accompanied by a softening of the response of the specimen.

The mechanisms responsible for the formation of shear bands vary widely from one material to another. However, a common feature of the processes of localization envisioned here is that they arise as a result of an instability in the inelastic behaviour of the solid. An explanation of this nature was first noted by Mandel and subsequently by Rudnicki and Rice for the inception of faulting in brittle rock masses under compression. A noteworthy outcome of their analysis is that in the absence of normality localization can take place, even under hardening conditions, and thus is not necessarily associated with softening in the material response. A comparison of bifurcation conditions and localization directions for various geomaterial models can be found in Reference 10. By contrast, a full understanding of post-bifurcation phenomena is yet to emerge. Remarkably, experiments suggest that localization may result in the overall softening of the specimen, even while the material remains hardening at all times. Such softening is thus structural or geometrical in nature and not attributable to the constitutive behaviour of the material, as noted by Drescher and Vardoulakis (see also the review by Read and Hegemier).

The question of whether softening in frictional solids is geometrical in nature or intrinsic to the material is one upon which much light can be shed by means of detailed numerical simulations. However, conventional finite elements may fail to provide any meaningful information about strain localization: we present an example in which a plane strain sand specimen is subjected to...
uniaxial compression, and for which isoparametric elements inhibit localization completely and yield no solutions other than the uniform state. Examples of this nature illustrate the need for specialized finite elements tailored to this class of problems.

A finite element method which overcomes some of the difficulties associated with conventional elements in problems involving strain localization was proposed by Ortiz et al.,\textsuperscript{14} and subsequently extended to the finite deformation range by Nacar et al.\textsuperscript{15} The method relies on the fact that the conditions for localization are purely local—that they can be ascertained in a pointwise fashion. Once the onset of localization is detected within an element, its strain interpolation may be enriched by means of additional shape functions which reproduce the emerging localized deformation patterns. The results of the computations are thus fairly independent of the relative orientation of the shear band with respect to the mesh. Furthermore, the band broadening which is frequently associated with isoparametric elements is eliminated, and the full resolution of the mesh is realized. Finally, the addition of the specialized modes significantly enhances the ability of the elements to shear without delaying the progression of geometrical softening.

In this paper, we examine the performance of the method, with particular emphasis on three-dimensional applications and materials with internal friction. It should be noted that other finite element methods employed in the past for the analysis of two-dimensional strain localization, such as the use of rectangular elements composed of four crossed triangles (see e.g. Reference 16) cannot easily be extended to three dimensions. This lack of generally applicable methods of analysis is partly responsible for the paucity of results on three-dimensional strain localization.

Throughout this work, we confine our attention to models of local continua. A limitation of this formulation is that, for rate-independent materials and in the absence of thermal effects, the thickness of the band does not follow directly from the field equations. In a finite element context, the band thickness is set by the mesh size. Thus, care must be exercised in designing the mesh in order to ensure that the relevant length scales of the problem are suitably resolved. Needleman\textsuperscript{17} has noted that rate-dependent constitutive formulations remove this indeterminacy by virtue of the fact that the band thickness is then set by the size of defects. Work by Molinari and Clifton\textsuperscript{18} also shows that, for metals undergoing high rates of straining, the band thickness is set by thermal softening coupled with heat conduction. Other authors (see e.g. References 19–25) have sought to overcome this difficulty by recourse to models of non-local continua, for which a characteristic length scale is built into the field equations. In the interest of simplicity, we choose here to ignore this possibility.

2. CONSTITUTIVE FRAMEWORK

The constitutive behaviour of a broad class of materials is characterized by pressure-sensitive yield followed by frictional sliding. The presence of internal friction renders the plastic flow non-associated. Sands, rocks and ceramics exhibit these effects as salient features of their constitutive behaviour. Also, processes of strain localization are common in these materials. It has been noted\textsuperscript{8,9} that lack of normality makes strain localization possible in the hardening range. Moreover, the pressure sensitivity of yield may have a marked influence on the post-bifurcation behaviour of the material. In the plane strain uniaxial compression test, for instance, the transition to shear flow within the band results in a characteristic ‘two-shelved’ softening behaviour of sand specimens.\textsuperscript{6,7}

A simple constitutive framework which incorporates pressure sensitivity and non-associated flow in a manner particularly well suited to computation is provided by the Drucker–Prager plasticity model.\textsuperscript{26} In particular, we focus our attention on yield criteria of the form

\[ F(\sigma, x) = Q + x(P - P_0) \leq 0 \]  \hspace{1cm} (1)

where

\[ Q \] are the respective Mohr–Coulomb

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where $\sigma$ is the Cauchy stress tensor and

$$
P = \frac{1}{2} \sigma_{ij},
$$

$$
Q = \sqrt{\frac{1}{2} S_{ij} S_{ij}}, \quad S_{ij} = \sigma_{ij} - P \delta_{ij}
$$

(2)

are the corresponding pressure, equivalent shear stress and deviatoric stress component, respectively. Furthermore, the elastic response is assumed to be linear. By matching the Mohr–Coulomb model in the triaxial test, the parameters $\alpha$ and $P_0$ in equation (1) are found to be related to the friction angle $\phi$ and cohesion $c$ through

$$
\alpha = \frac{6 \sin \phi}{3 - \sin \phi}, \quad c = P_0 \tan \phi
$$

(3)

Finally, the plastic flow of the material is assumed to be rate-independent and to obey a flow rule of the type

$$
\dot{\epsilon}^P_{ij} = \frac{1}{2} \frac{\partial G(\sigma)}{\partial \sigma_{ij}}
$$

(4)

where the flow potential $G$ takes the form

$$
G(\sigma, \beta) = Q + \beta P
$$

(5)

By analogy with equation (3), the parameter $\beta$ in equation (6) can be expressed in terms of a dilatancy angle $\psi$ as

$$
\beta = \frac{6 \sin \psi}{3 - \sin \psi}
$$

(6)

It is noted that normality is lost for values of $\beta \neq \alpha$. In the above constitutive framework, the number of plastic parameters reduces to three: namely, $P_0$, $\alpha$ and $\beta$. This renders parametric studies particularly simple.

A final issue concerns the modelling of plastic hardening. Here we simply assume the friction angle $\phi$ to be a function of an effective plastic strain, whose rate is defined as

$$
\dot{\epsilon}^P = \sqrt{\frac{1}{2} \dot{\epsilon}_{ij}^P \dot{\epsilon}_{ij}^P}
$$

(7)

The dilatancy angle is assumed to remain constant. The simple functional dependence,

$$
\sin \phi = \sin \phi_i + \frac{k}{\dot{\epsilon}^P} + \frac{k}{\dot{\epsilon}^P}, \quad k = 2(\sin \phi_i - \sin \phi) \sqrt{\dot{\epsilon}^P}
$$

(8)

is adopted here. This form is similar to a model used by de Borst. Equation (8) establishes a smooth transition of $\phi$ from an initial value $\phi_i$ to a maximum $\phi_f$ attained when the effective plastic strain reaches a critical value $\dot{\epsilon}^P_c$.

### 3. LOCALIZATION IN FRICTIONAL MATERIALS

#### 3.1. Continuum basis

We start by reviewing selected results on localization which bear directly on the discussions that follows. A comprehensive discussion of the subject is given by Rice; here, attention is confined to small deformations and thermally decoupled, rate-independent material behaviour.
We envision a solid undergoing homogeneous deformation and seek to determine conditions under which an alternative mode of deformation exhibiting a plane of strain discontinuity becomes possible. Let \( \mathbf{n} \) denote the normal to the incipient plane of discontinuity. Then, Maxwell's compatibility conditions restrict the jump in the velocity gradients to be of the form
\[
[\dot{u}_{i,j}] = \dot{u}_{i,j}^{+} - \dot{u}_{i,j}^{-} = \hat{g}_{i} n_{j}
\]
where the superscripts 'plus' and 'minus' refer to the two sides of the plane of discontinuity, regarded as an oriented surface, and \( g_{i} \) are arbitrary. It proves convenient to introduce the unit vector
\[
m_{i} = \dot{g}_{i} / \| \dot{g} \|, \quad \hat{g} = \| \dot{g} \|
\]
which may be interpreted as defining the direction of relative displacement of points on the plus side of the surface of discontinuity with respect to points on the minus side. With this notation, equation (9) becomes
\[
[\dot{u}_{i,j}] = \hat{g} m_{i} n_{j}
\]
Likewise, the strain field exhibits a jump of the form
\[
\left[ \varepsilon_{i,j} \right] = \frac{1}{2} (\hat{g} n_{i,j} + n_{i} \hat{g}_{j}) = \frac{1}{2} \hat{g} (m_{i,j} + m_{i} n_{j})
\]
Equilibrium across the discontinuity requires the tractions to be continuous:
\[
\left[ t \right] = \mathbf{n} \cdot \left[ \varepsilon \right] = 0
\]
For rate-independent solids the incremental stress-strain relations take the form
\[
\dot{\sigma} = \mathbf{D} : \dot{\varepsilon}
\]
where \( \mathbf{D} \) is the tangent stiffness of the material. For plastic solids, \( \mathbf{D} \) has two branches, corresponding to plastic loading and elastic unloading, respectively. It is known from plastic stability theory\(^{30} \) that, in solids obeying the normality law, the earliest possible bifurcation point is obtained by assuming the plastic loading branch of \( \mathbf{D} \) on both sides of the discontinuity, despite the fact that one side may unload elastically immediately following bifurcation. This corresponds to investigating the stability of Hill's linear comparison solid.\(^{31} \) In materials exhibiting lack of normality, the linear comparison solid may not always yield the lowest bifurcation point.\(^{32} \) However, it is frequently found that it provides a close approximation.\(^{33} \) Adopting Hill's procedure, equations (13) and (14) can be combined to yield
\[
\mathbf{n} \cdot \mathbf{D} : \left[ \dot{\varepsilon} \right] = 0
\]
where \( \mathbf{D} \) is the plastic branch of the plastic modulus. Finally, bringing in the kinematic condition (11), the above becomes
\[
A_{p} (n) m_{i} = 0
\]
where
\[
A_{p} (n) = D_{i,k,l} n_{i} n_{k}
\]
Clearly, equation (16) requires that
\[
\det (A(n)) = 0
\]
Thus, a necessary condition for localization is that equation (18) have solutions \( n \). Such solutions determine the normal to the possible planes of strain discontinuity. The geometry of the localized
mode is fully determined by \( n \) and the corresponding zero eigenvectors \( m \) of the acoustic tensor (17). Following the pioneering work by Mandel, Rudnicki and Rice applied the above methodology to frictional materials. A noteworthy outcome of their analysis is that lack of normality in the plastic flow may result in localization even in the presence of hardening. More recently, Vardoulakis has conducted an experimental investigation of the behaviour of dry sands in plane strain. Based on the Mohr–Coulomb model of soil plasticity, he found that the inclination \( \theta \) of the shear band relative to the principal stress axes is given by the expression

\[
\theta = \frac{1}{2} \pi + \frac{1}{4} (\phi_0 + \psi_b)
\]  

(19)

first proposed by Arthur et al., where \( \phi_0 \) and \( \psi_b \) are the values of the friction and dilatancy angles at the onset of bifurcation.

3.2. Finite element implementation

A salient feature of the localization condition is that it is purely local—that it can be elucidated in a pointwise fashion. In a finite element context this implies that localization can be monitored entirely at the element level. In this section we illustrate the details of this procedure by way of a simple one-element example.

Typical stress–strain curves for the material model outlined in Section 2 are shown in Figure 1. The curves correspond to plane strain uniaxial compression with \( E = 2 \times 10^{5}, \nu = 0.2, P_0 = 2 \times 10^{5} \), and were generated from the one-element model inset in Figure 1 and by a step-by-step integration of the constitutive relations. During the integration procedure, the localization condition (16) can be checked at selected points within the element. In general, stresses in isoparametric elements are most accurately computed at the reduced quadrature points, which thus become the optimal sampling points for localization as well. In the present example, this distinction is superfluous since the state of stress is uniform throughout the element.

![Figure 1. Plane strain compression stress–strain curves for the material model adopted in the numerical simulations. The onset of bifurcation is indicated on each curve by a circle.](image-url)
Figure 2 depicts the value of the determinant \( \text{det} \left( \mathbf{A}(\mathbf{n}) \right) \) as a function of the angle \( \theta \) subtended by a prospective shear band relative to the axis of loading. Thus, \( \mathbf{n} = (-\sin \theta, \cos \theta) \). As may be seen, the determinant remains positive everywhere for sufficiently small values of \( \varepsilon_{22} \). This can be also appreciated from Figure 3, which depicts the evolution of the minima of the determinant with increasing strain. At a certain point, the determinant vanishes for the first time at values of \( \theta \) ostensibly similar to those predicted by formula (19). These values of \( \theta \) determine the directions along which incipient shear banding becomes possible. The points on the stress–strain curves at which the onset of localization occurs are shown in Figure 1. It is interesting to note that, for

![Figure 2. Evolution of the determinant of the acoustic tensor during the plane strain compression test up to localization](image)

![Figure 3. Evolution of the minima of the determinant of the acoustic tensor during the plane strain compression test](image)
choices of the parameters resulting in lack of normality, localization occurs on the rising part of the curves, in accordance with the observations by Rudnicki and Rice.\(^9\) In fact, all the eigenvalues of the element remain positive well beyond localization (Figure 4). Thus, localization in this class of frictional solids is not necessarily associated with softening constitutive behaviour.

To summarize: localization instabilities can be detected at the element level on the basis of information which becomes readily available during a typical incremental analysis. The algorithmic aspects of the computation of the minima of the determinant \(\det(A(n))\) are discussed in Appendix I. It is shown that in two dimensions the localization check reduces to computing the roots of a quartic polynomial. In three dimensions, the minima are computed iteratively.

4. ADDED-MODE ELEMENTS FOR LOCALIZATION ANALYSIS

The performance of conventional finite elements deteriorates rapidly in the presence of localization. Owing to the smooth character of the interpolation, isoparametric elements are incapable of developing strain jumps of arbitrary orientation. As a result, localization may be delayed or precluded altogether, shear bands broadened beyond the resolution of the mesh and the accompanying geometrical softening retarded. The performance of these classes of elements can be substantially improved by suitably enriching the local strain interpolation.\(^{14,15}\) In this section, a brief summary of the main ideas involved in the method is given.

4.1. Localized deformation modes

We consider general isoparametric elements for which the local displacement fields are expressible in the form

\[
    u_i(x) = \sum_u u_{ia} N_a(x)
\]  

(20)
where \( u_{ia} \) are the nodal displacements and \( N_a \) the corresponding shape functions, and the sum extends over all nodes in the element. Furthermore, we shall use the symbol \( \xi_a \) to denote the location of the \( x \)th reduced quadrature point. For instance, four-node quadrilaterals and eight-node bricks contain one single reduced quadrature point coincident with the centroid of the element.

Next, we envision an incremental finite element analysis of a rate-independent inelastic body undergoing small strains. As discussed in Section 3.2, at each stage of the analysis we can monitor localization at, say, the reduced quadrature points within each element. Let us assume that at some point localization becomes possible at location \( \xi_a \) in some element of the mesh. The localization analysis itself then provides the direction \( n_a \) of the incipient plane of strain discontinuity, as well as the remaining characteristic direction \( m_a \), which completes the description of the kinematics of the localized mode (equation (12)).

At this point we replace the original interpolation of the element (20) by the enriched displacement field

\[
 u_i(x) = \sum u_{ia} N_a(x) + \sum g_x M_{ia}(x) \tag{21} 
\]

where the second summation extends to all the active localized modes within the element. The specialized shape functions \( M_{ia} \) are set up so as to mimic the incipient localized deformation modes while preserving the satisfaction of the patch test. A choice consistent with these constraints\textsuperscript{14, 36} is

\[
 M_{ia}(x) = \chi_{ia}(x) m_{ia} n_a (x_l - \xi_{ia}) \quad \text{(no sum on } a) \tag{22} 
\]

where

\[
 \chi_{ia}(x) = \begin{cases} 
 A_+ / A, & \text{if } (x_l - \xi_{ia}) m_{ia} > 0 \\
 -A_- / A, & \text{if } (x_l - \xi_{ia}) n_{ia} < 0 
\end{cases} \tag{23} 
\]

Here \( A_+ \) and \( A_- \) are the areas of the plus and minus parts of the element, and \( A = A_+ + A_- \).

A minor computation\textsuperscript{14} shows that equation (21) results in strain fields exhibiting jumps of the type (9). It should be noted that the addition of the localized modes renders the element incompatible. However, the choice of weights in equation (22) ensures satisfaction of the patch test, and thus the enriched element may be expected to produce convergent approximations.\textsuperscript{37} Finally, we note that the amplitudes of the localized modes constitute unknowns to be determined as part of the analysis. However, these additional degrees of freedom are internal to the elements and can therefore be eliminated locally by means of static condensation. Further details may be found in Reference 14.

5. NUMERICAL EXAMPLES

The examples discussed in this section are intended to bring out the differences in performance between isoparametric and added-mode elements, as well as the inability of the former to handle localization adequately. In passing, we show how localization may account for the softening which is observed in the plane strain compression test for sands. The constitutive framework outlined in Section 2 is adopted throughout the calculations.

5.1. Plane strain compression test

Our first example is concerned with plane strain specimens of cohesionless sand subjected to biaxial compression. Experimental and analytical results for this problem have been given by
Vardoulakis and co-workers\textsuperscript{1,2} numerical simulations may be found in Reference 35. Both analysis and experiment point to the development of sharp shear bands accompanied by a marked softening of the specimen. Eventually, the force–displacement curve stabilizes to a lower plateau.\textsuperscript{6}

The localized modes discussed in Section 4 were introduced as soon as an element yielded, and were oriented along the minima of the determinant of the tangent modulus. It should be noted that, in so doing, the localized modes are generally introduced before localization properly occurs. In this case, the determinant may be interpreted as giving a measure of the tendency to localization, and its minima as precursors of the likely characteristic directions. From a numerical standpoint, it was found that the introduction of the specialized modes at the earliest possible stage made for a smoother transition into the fully developed shear band regime. It was also found by numerical testing that the best results were obtained when the characteristic directions were updated every few steps. In the calculations discussed here, the characteristic directions were renewed every five increments.

In our calculations we have chosen the dilatancy angle to be zero, with a view to simulating the critical state conditions at which plastic flow takes place at constant volume. It was found that plastic dilatancy promoted bulging instabilities, whereas isochoric plastic flow was more conductive to shear banding. Four-node quadrilateral elements were used throughout the calculations. To prevent mesh locking as a result of near-incompressibility, we adopt Hughes's $\overline{B}$-method,\textsuperscript{38} which consists of extrapolating the volumetric strains from the centroid of the element. The confining pressure, which is assumed to remain constant throughout the test, was built into the calculations by endowing the material with a fictitious cohesion. An imperfection in the form of a soft element was introduced at various locations within the specimen to act as a nucleation site for localization. Longitudinal displacements were prescribed at both ends of the specimen so as to simulate rigid frictionless plates.

A first example (Figures 5–9) is concerned with a specimen with an imperfection at the centre. The material was assumed to have a Young’s modulus $E=2 \times 10^9$, a Poisson ratio $\nu=0.25$, a dilatancy angle $\psi=0^\circ$, an initial friction angle $\phi_i=10^\circ$, and a saturation friction angle $\phi_s=20^\circ$.

![Figure 5. Load-displacement curve for the symmetric plane strain compression test. The circle indicates the onset of bifurcation in the homogeneous solution](image-url)
reached at a plastic strain $\varepsilon_p = 0.5$ per cent. The confining pressure was taken to be $P_0 = 2 \times 10^5$. The soft element was assumed to have a constant friction angle of $10^\circ$ (i.e. $\phi_r = \phi_t = 10^\circ$). A second example (Figures 10–14) concerns a specimen with an imperfection at the lower left corner. The material properties adopted in the second analysis were identical to those of the first, except that $\phi_r$ was taken to be $22^\circ$ at the imperfection and $25^\circ$ throughout the rest of the specimen.

A comparison between the load–displacement curves computed from $\bar{B}$ and from the enhanced elements is shown in Figures 5 and 10. Both methods yield ostensibly identical results up to
Figure 8. Contours of maximum shear strain for the symmetric plane strain compression test as computed by the added-mode method. There are 10 equally spaced contours ranging from 0.33 to 1.50 per cent.

Figure 9. Distribution of effective plastic strain along cuts through the shear band for the symmetric plane strain compression test. Comparison of results from isoparametric and enhanced elements.

Localization. However, their behaviour is in sharp contrast once shear banding sets in. Thus, the conventional element yields what is roughly the uniform solution. This may be appreciated from Figures 5 and 10 by a comparison with the stress–strain curve of the material, which is included for reference. Also, the effective plastic strain is plotted in Figures 9 and 14 along cuts transverse to the shear band. It is again observed how the isoparametric element produces the uniform solution and precludes strain localization of any appreciable magnitude. Finally, it was found that the
isoparametric elements did not result in any elastic unloading anywhere in the mesh in the case of the central imperfection. For the corner inhomogeneity, unloading was confined to the upper portion of the specimen.

This provides good evidence of the inability of conventional methods to deal adequately with problems of strain localization. In particular, it is seen that the isoparametric element inhibits localization altogether. It should be emphasized that the results reported here may not be indicative of any deficiencies inherent to use of the $\bar{B}$ element. In fact, this method seems to be quite
Figure 12. Deformed mesh for the unsymmetric plane strain compression test as computed by the added-mode method. Displacements are magnified by a factor of 45.

Figure 13. Contours of maximum shear strain for the unsymmetric plane strain compression test as computed by the added-mode method. There are 10 equally spaced contours ranging from 0.38 to 1.10 per cent.

effective in alleviating mesh locking as a result of near-incompressibility. However, the difficulties encountered in the present example are associated not with incompressibility but rather with strain localization. Thus, for instance, a slope instability example was analysed by Ortiz et al. in which the soil was taken to be plastically dilatant, so that incompressibility was of no concern. There, too, conventional elements were found to be beset by the same problems alluded to here.
Figure 14. Distribution of effective plastic strain along cuts through the shear band for the unsymmetric plane strain compression test. Comparison of results from isoparametric and enhanced elements.

By way of contrast, the enhanced element does produce a sharp shear band (Figures 8 and 13) and a steep descending branch in the load–displacement curve (Figures 5 and 10), in agreement with experimental evidence. The vast discrepancies in the results of the two methods are also apparent in the plots of effective plastic strain (Figures 9 and 14), where the enhanced element is seen to result in the formation of sharp shear bands. Similar conclusions may be drawn from Figures 7 and 12, which show the deformed mesh at the end of the computations. As may be seen, the deformation is confined to a narrow band of material, while the remaining parts of the mesh essentially experience rigid body motions. Finally, Figures 6 and 10 depict the bifurcated elements and corresponding characteristic directions which, as expected, closely trace the shear bands. The process of localization was found to be accompanied by general unloading outside the shear band.

5.2. Discussion

The above computations lend support to the conjecture that the softening which is observed in plane strain biaxial sand specimens may result entirely from geometrical effects and not from material behaviour. Thus, while the constitutive laws adopted in the analysis assumed the material to remain hardening at all times, the resulting force–displacement diagrams exhibited marked softening as a consequence of localization. That localization is possible in hardening materials lacking normality was shown by Mandel and Rudnicki and Rice. Their analysis, however, did not extend to the post-bifurcation regime.

A further striking feature of the force–displacement curves measured in plane strain biaxial compression is the existence of an asymptotic lower plateau following the softening regime. A simple analytical argument which provides some insight into this phenomenon and into the numerical results discussed above is the following. For the variant of the Drucker–Prager model described in Section 2, the maximum stress in plane strain uniaxial compression is readily found to be

\[ P_h = 2P_0 \sqrt{3} / (\sqrt{3} - \alpha) \]  

(24)
where the label ‘h’ refers to the fact that the stress $P_h$ provides a measure of the load-carrying capacity of the soil under conditions of homogeneous deformation. The value of $P_h$ for the material parameters adopted in the numerical examples is shown in Figures 5 and 10. Close agreement is observed between the analytical prediction and the computed maximum stress.

As localization sets in, however, a rather abrupt transition takes place from a uniform mode of deformation to a regime of predominantly shear action within the band. For frictional materials of the type considered here, the shear strength is considerably lower than the compressive plane strain strength, owing to the larger confinement experienced by the solid in the latter case. A simple model which allows one to quantify the loss of bearing capacity due to localization is obtained by considering an infinite body traversed by an infinite shear band. Similar idealizations have been found useful in the study of localization in metals. Here, the orientation of the shear band is found from the bifurcation analysis outlined in Section 3.1. A steady state is assumed to ensue in which the material in the band flows plastically at constant volume while the surrounding material unloads elastically. A straightforward analysis (given in Appendix II) yields the expression

$$P_h = \frac{P_0 \alpha}{\sqrt{3 \sin \theta - \alpha \cos \theta} \cos \theta}$$

(25)

for the remote compressive stress. The value $P_h$ may be envisioned as an estimate of the stress borne by the specimen when the shear band is fully developed—that is, as the level of the lower shelf in the load–displacement curve. Note that by virtue of the simplifications adopted, neither the size of the specimen nor the thickness of the shear band have any bearing on the estimate (25).

The value of $P_h$ for the material parameters used in the numerical simulations is shown in Figures 5 and 10. As may be seen, (25) overestimates the level of the lower plateau. A more accurate estimate is likely to necessitate consideration of the finiteness of the specimen as well as of the conditions prevailing in the vicinity of the imperfection from which the band nucleates. Thus, our numerical results exhibit a sharp decrease in mean stress which starts at the imperfection and propagates along the band. This in turn results in an abrupt reduction in equivalent shear stress, with the concomitant loss of bearing capacity of the specimen. Of particular significance to the present discussion, however, is the fact that

$$P_h < P_h$$

(26)

which accounts for the upper and lower limit loads observed in experiments. Thus, following the onset of localization the body steadily loses a considerable portion of its load-carrying capacity as the shear band develops fully. As a result, the load–displacement curve is dragged down towards the lower shelf, as suggested by our numerical experiments (Figures 5 and 10). It is interesting to note that this type of behaviour is characteristic of frictional materials and is not observed in metals.16

5.3. Three-dimensional slope stability problem

Our last example concerns a slope stability problem in three dimensions. A quarter of the embankment considered in the analysis is shown in Figure 15, where the vertical faces are taken to be planes of symmetry. The remaining lateral surfaces subtend a 1:1 slope. The embankment rests on a rigid foundation and was subjected to increasing gravity loads. No relative displacements between the embankment and the foundation were allowed. This example is a three-dimensional counterpart of a similar plane problem discussed by Ortiz et al.14

Figure 16 shows a finite element discretization of the embankment into eight-node brick elements. The analysis was conducted for a Young’s modulus $E = 2 \times 10^6$, a Poisson’s ratio $\nu = 0.25$
and a constant friction angle $\phi_i = \phi_e = 20^\circ$. Thus, the material was assumed to exhibit elastic/perfectly plastic behaviour. The dilatancy angle was chosen to be 10°, resulting in dilatant plastic flow under fully plastic conditions. Hence, mesh locking as a result of near-incompressibility is of no concern, and simple isoparametric elements may be used in the analysis.

The computed force–displacement curves are shown in Figure 16. Whereas the enhanced element predicts a limiting load, as befits perfect plasticity, the isoparametric elements produce a steadily rising curve. Here again, the best results were obtained by repeating the local bifurcation analysis every few increments in order to update the characteristic directions.
A plot of the final deformed mesh, as computed from the enhanced elements, is shown in Figure 17. A failure mechanism by sliding of the slopes is clearly apparent. Figures 18 and 19 depict the corresponding level contours of effective plastic strain. Two sets of three-dimensional, doubly curved shear surfaces are apparent. One set results in the sliding of the inclined edges of the embankment; the slopes fail along the other set. The shear surfaces intersect pairwise (Figure 18), giving rise to an intricate pattern of interacting failure modes. The trace of the contours of effective plastic strain on the planes of symmetry constitutes a pattern which is strongly reminiscent of the plane strain solution discussed by Ortiz et al.14

A principal objective in considering this example was to test the ability of the method to deal with three-dimensional problems effectively. It should be noted that methods such as the use of
cross-triangular meshes, which have been successfully used in plane problems, cannot be readily generalized to three dimensions. The added-mode technique also appears to be quite effective at eliminating spurious dependences of the solution on the orientation of the mesh. In particular, the method is seen to facilitate the development of intricate patterns of doubly curved shear surfaces intersecting the mesh at arbitrary angles.

APPENDIX I

Computation of the characteristic directions

For completeness, here we summarize two algorithms for computing the characteristic directions \( \mathbf{m} \) and \( \mathbf{n} \) introduced in Section 3.1. For convenience, we treat the two-dimensional and three-dimensional cases separately.

**Three-dimensional case.** Consider the following constrained minimization problem:

\[
\begin{align*}
\text{minimize} \quad & f(\mathbf{n}) = \det(n_i D_{ijkl} n_j) \\
\text{subject to} \quad & |\mathbf{n}| = 1
\end{align*}
\]

(27)

where \( \mathbf{D} \) is the current value of the tangent moduli. The minima occur at the values of \( \mathbf{n} \) for which the determinant (18) is closest to vanishing; localization becomes possible when one of these minima becomes zero. The solutions of the problem (27) are characterized by the condition

\[
\frac{\partial}{\partial n_i} \left[ f(\mathbf{n}) - \lambda |\mathbf{n}|^2 \right] = \frac{\partial f(\mathbf{n})}{\partial n_i} - 2\lambda n_i = 0
\]

(28)

where \( \lambda \) is a Lagrange multiplier. Differentiation of \( f \) in equation (28) yields

\[
\det(\mathbf{A}(\mathbf{n})) D_{ijkl} A^{-1}_{ij} (\mathbf{n}) n_i - \lambda n_i = 0
\]

(29)
Letting

\[ J_i(n) = \det(A(n)) D_{ijkl} A_{ikl}^{-1}(n) \]  

the condition (29) may be restated as

\[ J_i(n)n_i - \lambda n_i = 0 \]  

The solutions of this equation and of the subsidiary condition \(|n| = 1\) may be found as follows:

1. Expressing \(n\) in terms of spherical angles (i.e. setting \(n = \cos \phi \cos \theta, \cos \phi \sin \theta, \sin \phi\)), the range of variation \([0, 2\pi] \times [0, \pi/2]\) of \((\theta, \phi)\) is swept at 5° increments to determine a first approximation \(n^{(0)}\) to the minima.

2. The locations of the minima are then refined iteratively according to the scheme

\[ J_i(n^{(k)})n_i^{(k+1)} - \lambda^{(k+1)} n_i^{(k+1)} = 0 \]  

Every iteration constitutes a three-dimensional eigenvalue problem. The convergence of the iterative procedure is quadratic. The remaining characteristic vector \(m\) is then determined as the zero eigenvector of the acoustic tensor \(A(n)\) given by equation (16).

Two-dimensional case. In two dimensions the determinant of the tangent moduli can be written explicitly as

\[ \det(A(n)) = a_0 n_1^4 + a_1 n_1^3 n_2 + a_2 n_1^2 n_2^2 + a_3 n_1 n_2^3 + a_4 n_2^4 \]  

where

\[
\begin{align*}
a_0 &= D_{1111}D_{1212} - D_{1112}D_{1211} \\
a_1 &= D_{1111}D_{1222} + D_{1112}D_{1212} - D_{1112}D_{2211} - D_{1122}D_{1211} \\
a_2 &= D_{1111}D_{2222} + D_{1112}D_{1222} + D_{1122}D_{2212} - D_{1122}D_{1212} - D_{1122}D_{2211} \\
a_3 &= D_{1112}D_{2222} + D_{1211}D_{2222} - D_{1122}D_{2212} - D_{1222}D_{2211} \\
a_4 &= D_{1212}D_{2222} - D_{1222}D_{2212}
\end{align*}
\]

As in three dimensions, we seek to determine the minima of equation (33). Such minima may be computed by letting \(n_1 = \cos \theta, n_2 = \sin \theta\) in equation (33) and requiring that

\[ \frac{d}{d\theta} \det(A(n)) = A_1 n_1^4 + A_2 n_1^3 n_2 + A_3 n_1^2 n_2^2 + A_4 n_1 n_2^3 + A_5 n_2^4 = 0 \]  

from which

\[ A_1 = a_1, \quad A_2 = -4a_0 + 2a_2, \quad A_3 = 3a_3 - 3a_1, \quad A_4 = 4a_4 - 2a_2, \quad A_5 = -a_3 \]  

Finally, letting \(x = \cotan(\theta)\), equation (35) reduces to

\[ A_1 x^4 + A_2 x^3 + A_3 x^2 + A_4 x + A_5 = 0 \]  

The roots of equation (37) may be found by using Ferrari’s formulae for the zeros of a quartic polynomial. As in three dimensions, the onset of localization is signalled by the vanishing of one or more of the minima. The acoustic tensor \(A(n)\) then becomes singular, and the characteristic vectors \(m\) follow as the corresponding zero eigenvectors.
Appendix II

Computation of limit loads

Here we seek to compute an estimate of the compressive load at which a shear band in plane strain uniaxial compression reaches a steady state. We idealize the body as being unbounded and traversed by a straight shear band of constant thickness. The orientation of the band is characterized by the angle $\theta$ subtended by its plane and the $X$ axis (Fig. 20). The body is acted upon by a remote uniaxial compression $-\vec{\sigma}$ along the $Y$ axis. The state of stress is assumed to be uniform inside the shear and in the surrounding region.

A local reference frame $(x, y)$ is defined such that $x$ points in the direction tangent to the band and $y$ in the direction of the normal. In this co-ordinate system, the stress tensor outside the band takes the form

$$\sigma_1 = \tilde{\sigma} \begin{pmatrix} \sin^2 \theta & \sin \theta \cos \theta \\ \sin \theta \cos \theta & \cos^2 \theta \end{pmatrix}$$

whereas inside the band

$$\sigma_2 = \begin{pmatrix} \sigma_x & \tau & 0 \\ \tau & \sigma_y & 0 \\ 0 & 0 & \sigma_z \end{pmatrix}$$

Equilibrium across the boundary of the band requires that the tractions be continuous:

$$\tau = -\tilde{\sigma} \sin \theta \cos \theta, \quad \sigma_y = -\tilde{\sigma} \cos^2 \theta$$

Maxwell's compatibility conditions, on the other hand, require that

$$[\tilde{e}] = \frac{1}{2} \begin{pmatrix} 0 & \dot{\gamma}_x \\ \dot{\gamma}_x & 2\dot{\gamma}_y \end{pmatrix}$$

for some as yet unknown vector $\dot{\gamma}$.

Figure 20. Idealized geometry utilized to estimate bearing capacity of plane strain biaxial specimen under conditions of fully developed shear banding.
ANALYSIS OF STRAIN LOCALIZATION

Next, we assume that the body has reached stationary conditions—that the stress rates vanish identically everywhere. As a consequence, the elastic strain rates must vanish identically as well. Therefore, if we assume that the material surrounding the band is deforming elastically, we reach the conclusion that it must in fact move rigidly. Let us further assume that the material inside the band is at the critical state and thus flows plastically at a constant volume. Under these conditions, one finds from the flow rule (4) that

\[ \dot{e}_x = \frac{3\lambda}{2Q} S_x = 0 \]  

(42)

However, the plane strain condition leads to

\[ \dot{e}_2 = \frac{3\lambda}{2Q} S_x = 0 \]  

(43)

from which we conclude that

\[ \sigma_x = \sigma_y = \sigma_z \]  

(44)

At this point we have five equations at our disposal, namely (44), the jump conditions (40) and the yield criterion (1), which together suffice to determine the five unknowns \( \sigma_x, \sigma_y, \tau, \sigma_z \) and \( \delta \). For instance, one can eliminate the stress components in favour of the stress invariants (2), which are found to take the form

\[ Q = \bar{\sigma} \sqrt{3} \sin \theta \cos \theta, \quad P = -\bar{\sigma} \cos^2 \theta \]  

(45)

Finally, making use of the yield criterion (1) we may determine the sought value of \( \bar{\sigma} \), which is

\[ \bar{\sigma} = \frac{P_0\sigma}{(\sqrt{3} \sin \theta - \alpha \cos \theta) \cos \theta} \]  

(46)

as a function of the orientation \( \theta \) of the band.

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