A METHOD OF HOMOGENIZATION OF ELASTIC MEDIA

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Abstract—A perturbation technique is proposed which provides a simple means of estimating the effective behavior and fluctuation fields of a heterogeneous elastic medium. The perturbation analysis is based on an integral equation which characterizes the Fourier transform of the fluctuation stress potential. The average stress tensor drives the microstructural response and is assumed given. In contrast to other perturbation methods, the first-order approximation provides a nontrivial correction to the Voigt average moduli and information concerning the strain fluctuations. The first-order term in the expansion follows from straightforward computations and incorporates the statistical information provided by two-point spatial correlations of the elastic properties. Closed form expressions are obtained for the effective moduli of a two-phase continuum with randomly distributed inclusions and the results compared against the predictions of other methods.

1. INTRODUCTION

The problem of estimating the overall properties of elastic media with heterogeneous microstructure has been the subject of extensive research dating back to the pioneering work by Voigt [1, 2], Reuss [3] and Taylor [4, 5]. In recent studies a central role has been played by the self-consistent method introduced by MacKenzie [6], Kroner [7], Budiansky [8] and Hill [9] and later refined by numerous authors (see [10, 11] for an extensive list of references). The method is based on a result by Eshelby [12] for an ellipsoidal inclusion in an unbounded elastic medium and has been successfully applied to solids containing voids, composites and polycrystalline materials.

The above methods are based on certain simplifying assumptions concerning the interaction between inclusions and the surrounding medium. An alternative approach is to depart from the field equations themselves and approximate the solution by means of a perturbation expansion. For example, Gubernatis et al. [13, 14] have used Green's function techniques to express the displacement solution in integral form. These integral equations are then taken as a basis for the perturbation analysis. A shortcoming of this method is that the first-order approximation merely produces the Voigt average moduli. Furthermore, to account for local strain fluctuations terms in the expansion beyond first-order are necessary. Frequently, the calculations involved in the computation of higher order terms are exceedingly cumbersome, which renders the method of limited applicability.

In this paper, an alternative perturbation technique is proposed which provides corrections to the Voigt average moduli and information concerning all fluctuation fields right from the first-order term in the expansion. This term follows from straightforward algebraic manipulations and incorporates the statistical information provided by two-point correlations. It has been noted by Beran [15] that results involving correlation functions higher than third-order are of doubtful practical value owing to the fact that higher order correlations are rarely known. Frequently, no microstructural statistical information is available beyond two-point correlations. Thus, it is of some practical importance to be able to construct approximations based on this level of information as simple a way as possible. From this standpoint, the method described below would appear advantageous with respect to previously proposed perturbation techniques.

In Section 2 the field equations of linear elasticity are reduced to a Fredholm integral equation of the second kind. To this end, the stress field is formulated as the sum of the prescribed average stress and the microscopic fluctuation stresses. Equilibrium is identically satisfied by expressing the latter in terms of a stress potential. Finally, the compatibility equations provide a set of partial differential equations for the fluctuation stress potential. Since the domain of the analysis can be idealized as being unbounded, Fourier transform
methods can be advantageously applied. This fact has been noted by several authors in the past [16, 17]. In Fourier space, the fluctuation stresses follow from a linear integral equation. An exact expression for the effective moduli can be formally given in terms of the inverse integral operator. However, the inversion of the integral equations presents formidable difficulties even for the simplest of microstructural arrangements. To make progress analytically, a perturbation or Neumann series is utilized. It is seen that the first-order approximant can be given in terms of a straightforward quadrature and provides nontrivial corrections to Voigt averages for both effective and fluctuation fields. Closed form expressions are obtained for the effective moduli of two-phase media with circularly symmetric correlation functions and the results compared with the predictions of other methods.

2. GENERAL CONSIDERATIONS

Here we consider a linear elastic body with two well-differentiated length scales. At the macrostructural level the medium can be idealized as being homogeneous. At the microscale, the elastic properties of the body exhibit a nonuniform spatial distribution. Let $\varepsilon_{ij}(x)$ and $\sigma_{ij}(x)$ be the microstructural strain and stress tensor fields and let $C_{ijkl}(x)$ denote the microstructural spatial distribution of elastic flexibility compliances, which are a given of the problem. The spatial average of a field is defined in the usual fashion. Thus, for instance, the average stresses are given by

$$\left\langle \sigma_{ij} \right\rangle \equiv \lim_{V \to \infty} \frac{1}{V} \int_V \sigma_{ij}(x) \, d^3x$$

(1)

At the macroscopic level, the entity $V$ plays the role of an infinitesimal material neighborhood whose state is described by average quantities [22]. From a microstructural perspective, $V$ plays the role of microcontinuum [23]. Passage to the macroscopic level is achieved through the averaging operation expressed in (1). Average quantities such as $\left\langle \sigma_{ij} \right\rangle$ are identified with macroscopic observables and are assumed to vary over characteristic lengths much larger than the microstructural length scale. This assumption makes it possible to choose the size of the microcontinua $V$ to be much larger than the microstructural length scale but much smaller than the characteristic length of variation of the macroscopic fields. Thus, although average state variables may exhibit a macroscopic spatial variation they can be regarded as uniform at the microscale. On the other hand, the fact that $V$ is much larger than the characteristic length scale of the microstructural fluctuations makes it possible to idealize the microcontinua as being unbounded. This justifies the passage to the limit indicated in eqn (1). A more precise rendition of these concepts can be found in [9].

The spatial fluctuation of a field is defined as the pointwise deviation from its average value. For instance, if we let $\delta\sigma_{ij}$ and $\delta\varepsilon_{ij}$ denote the fluctuation stress and strain fields, respectively, it follows at once that

$$\sigma_{ij} = \left\langle \sigma_{ij} \right\rangle + \delta\sigma_{ij} \quad \varepsilon_{ij} = \left\langle \varepsilon_{ij} \right\rangle + \delta\varepsilon_{ij}$$

(2)

Throughout the analysis that follows, it is assumed that the average stresses (1) take prescribed values. A first step towards characterizing the effective moduli is to determine the microscopic stress and strain fields consistent with a given distribution of elastic properties $C(x)$ and a prescribed value of the average stresses. Assuming linear elastic behavior, the equations to be satisfied by such fields are the following:

(i) Compatibility:

$$(V \times \varepsilon \times V)_{ij} = \varepsilon_{mn,k} \varepsilon_{lm,n} \varepsilon_{mn} = 0$$

(3)

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(ii) Equilibrium:

\[ (V \cdot \sigma)_{ii} = \sigma_{ii} = 0 \]  

(4)

(iii) Hooke's law:

\[ \varepsilon_{ij} = C_{ijkl} \sigma_{kl} \]  

(5)

A closely related question is whether there exists an effective flexibility tensor \( C_{ijkl} \) such that the averages \( \langle \varepsilon_{ij} \rangle \) and \( \langle \sigma_{ij} \rangle \) of the solutions \( \varepsilon_{ij} \) and \( \sigma_{ij} \) of partial differential equations (3), (4) and (5) satisfy

\[ \langle \varepsilon_{ij} \rangle = C_{ijkl} \langle \sigma_{kl} \rangle \]  

(6)

To elucidate this point, let us start by taking averages on both sides of (5) to obtain

\[ \langle \varepsilon_{ij} \rangle = \langle C_{ijkl} \sigma_{kl} \rangle \]  

(7)

We next recall a standard result from Fourier analysis which relates the average value of the product of two functions to their respective Fourier transforms,

\[ \langle fg \rangle = \lim_{V \to \infty} \frac{1}{V} \left\{ \frac{1}{(2\pi)^3} \int \hat{f}(k) \hat{g}(k) d^3k \right\} \]  

(8)

A summary of basic facts concerning the Fourier transform which are used in the derivations is given in the Appendix. Here, as in eqn (1), we envision a process whereby \( f \) and \( g \) are restricted to a closed compact subset \( V \) of \( \mathbb{R}^3 \) containing the origin. It is implicitly understood in (8) that \( \hat{f} \) and \( \hat{g} \) are the Fourier transforms of such restrictions. The limit in which \( V \) is allowed to become unboundedly large in all directions yields the average \( \langle fg \rangle \). Throughout the rest of this paper, the factor \( 1/V \) is retained for dimensional reasons but the limiting process indicated in (8) will be left implied. Applying (8) to the case at hand we find

\[ \langle \varepsilon_{ij} \rangle = \lim_{V \to \infty} \frac{1}{V} \left\{ \frac{1}{(2\pi)^3} \int \hat{C}_{ijkl}(k) \hat{\sigma}_{kl}(k) d^3k \right\} \]  

(9)

To solve equations (3), (4) and (5) in a way that can be conveniently combined with (9) one can proceed as follows. The fact that \( \sigma \) is divergence-free, as required by (4), implies the existence of a stress potential \( \chi_{ij} (x) \) such that (24)

\[ \delta \sigma_{ij} = (V \times \chi \times V)_{ij} = \chi_{mn,k} e_{ilm} e_{jin} \]  

(10)

Substituting this expression into the remaining field equations (3) and (5) the following partial differential equation for the stress potential is obtained

\[ V \times [C(x)(V \times \chi(x) \times V)] \times V + V \times (C(x)\langle \sigma \rangle) \times V = 0 \]  

(11)

Note that in this equation the average stresses \( \langle \sigma \rangle \) take prescribed values. The stress potential is also subject to the condition

\[ V \times \chi \times V = 0 \]  

(12)
expressing that the average of the fluctuation stresses must vanish. Taking the Fourier transform of (11) results in the following integral equation in Fourier space

$$
\frac{1}{(2\pi)^3} \int K_{ijkl}(\mathbf{k};\mathbf{k}') \xi_{kl}(\mathbf{k}') d^3 k' = N_{ijkl}(\mathbf{k}) \langle \sigma_{ki} \rangle
$$  \hspace{1cm} (13)

where the kernel $K$ and the tensor $N$ take the form

$$
K_{ijkl}(\mathbf{k};\mathbf{k}') = e_{irp} e_{jsq} e_{kmn} k_i k_j k_m k_n \tilde{C}_{pqmn}(\mathbf{k} - \mathbf{k}')
$$

$$
N_{ijkl}(\mathbf{k}) = e_{irp} e_{jsq} k_i k_j \tilde{C}_{pqkl}(\mathbf{k})
$$  \hspace{1cm} (14)

Equation (13) is a Fredholm integral equation of the first kind. Since $C(x)$ is a real function, it follows that $\tilde{C}_{ijkl}(\mathbf{k}) = \tilde{C}_{ijkl}(-\mathbf{k})$, which combined with the symmetries of $C$, namely $C_{ijkl} = C_{klij} = C_{ijlk} = C_{ijkl}$, implies that the operator defined by kernel $K$ is hermitian. From the decay properties at infinity of Fourier transforms it follows that

$$
\left[ \iint \|K(\mathbf{k};\mathbf{k}')\|^2 d^3 k d^3 k' \right]^{1/2} < \infty
$$  \hspace{1cm} (15)

and thus the kernel $K$ defines a Hilbert–Schmidt integral operator.

However, such integral operator is singular, owing to the fact that condition (10) determines $\chi$ only up to arbitrary compatible fields. In other words, the fluctuation stress field is left unchanged by gauge transformations of the type $\chi \rightarrow \chi + V^3 \phi$, where $\phi$ is an arbitrary vector field and $V^3 \phi = (\phi_{iji} + \phi_{ijj})/2$. To resolve this indeterminacy, one can impose a subsidiary or gauge condition on $\chi$ such as

$$
(V \cdot \chi)_{j} = \chi_{iji} = 0
$$  \hspace{1cm} (16)

When its domain of definition is restricted to stress potentials satisfying gauge condition (16), integral operator (13) becomes nonsingular and can be inverted by analytic methods such as the use of Neumann series or numerically. Let $H$ denote the kernel of the inverse operator. Then eqn (13) can be solved for $\chi$, which reads

$$
\tilde{\chi}_{ij}(\mathbf{k}) = \left[ (2\pi)^3 \int H_{ijkl}(\mathbf{k};\mathbf{k}') N_{klnm}(\mathbf{k}) d^3 k' \right] \langle \sigma_{mn} \rangle
$$  \hspace{1cm} (17)

Using the Fourier transforms of (2a) and (10) and identity (8) one readily obtains from (17)

$$
\langle \epsilon_{ij} \rangle = \tilde{C}_{ijkl} \langle \sigma_{kl} \rangle
$$  \hspace{1cm} (18)

where the effective flexibility compliances $\tilde{C}_{ijkl}$ are given by

$$
\tilde{C}_{ijkl} = \langle C_{ijkl} \rangle - \frac{1}{V} \iint N_{mnij}(\mathbf{k}) H_{mnpq}(\mathbf{k};\mathbf{k}') N_{pqkl}(\mathbf{k}) d^3 k d^3 k'
$$  \hspace{1cm} (19)

At this point, it is interesting to note that the overall elastic properties of the homogenized continuum are those of a simple elastic body with pointwise elastic compliances $\tilde{C}$. From eqn (19) it follows that $\tilde{C}_{ijkl} = \tilde{C}_{ikjl} = \tilde{C}_{ijlk} = \tilde{C}_{ijkl}$ and thus (18) derives from a strain energy potential. Beran and McCoy [18] and Levin [19] have argued that the average stress and strain tensors are related by a nonlocal law of the type

$$
\langle \sigma_{ij} \rangle(x) = \int L_{ijkl}(x,x') \langle \sigma_{kl} \rangle(x') d^3 x'
$$  \hspace{1cm} (20)

where one w
A multipolar representation is obtained by performing a Taylor expansion about \( x \) [18], from which it follows that (20) can be approximated by

\[
\langle \sigma_{ij} \rangle(x) = \tilde{D}_{ijkl}(x)\langle \varepsilon_{kl} \rangle(x) + \tilde{D}_{ijkl\mu}(x)\langle \varepsilon_{kl\mu} \rangle(x) + \cdots
\]  

(21)

If all terms in (21) but the first are neglected one obtains

\[
\langle \sigma_{ij} \rangle(x) = \tilde{D}_{ijkl}(x)\langle \varepsilon_{kl} \rangle(x)
\]  

(22)

where \( \tilde{D} = C^{-1} \) are the effective stiffness compliances of the material. Thus, (18) and (22) are portrayed in [18] as first-order approximations to the nonlinear behavior. However, it should be emphasized that in deriving (18) and (19) no approximations of any type have been resorted to. Thus, it is concluded that the effective behavior of the medium does indeed correspond to that of a simple continuum and is exactly described by eqns (18) and (22).

In cases for which the distribution of elastic properties follows a simple pattern, eqn (13) can be solved in close form and the corresponding effective moduli (19) computed exactly. Frequently, however, the microstructure of the material is random, and the distribution of elastic properties cannot be described analytically in simple terms. In these cases, a deterministic approach based on exact solution of integral equation (13) is impracticable since detailed information concerning the microstructure of the material is rarely available. A method for circumventing this difficulty is discussed next.

3. PERTURBATION SOLUTIONS

The fluctuation stress field and effective moduli determined by (13) and (19) can be approximated to any degree of accuracy by means of a perturbation or Neumann series expansion. In the present context, this method of approximation has the desirable property that it gradually incorporates into successive iterations statistical data concerning the microstructure of the material. Thus, the first-order term can be expressed in terms of the average elastic properties and their two-point correlations alone. Higher order approximations incorporate higher order spatial correlations.

In order to introduce the method, it proves convenient to rephrase (13) as a Fredholm integral equation of the second kind. To this end, let us express the microscopic elastic compliances \( C(x) \) as the sum of their average value and a fluctuation term, i.e.

\[
C_{ijkl}(x) = \langle C_{ijkl} \rangle + \delta C_{ijkl}(x)
\]  

(23)

In Fourier space this becomes

\[
\hat{C}_{ijkl}(k) = (2\pi)^3 \langle C_{ijkl} \rangle \delta(k) + \hat{\delta C}_{ijkl}(k)
\]  

(24)

where \( \delta(k) \) signifies the Dirac delta. Substituting this expression into (14), integral equation (13) takes the form

\[
\frac{1}{(2\pi)^3} \int \delta K_{ijkl}(k;k') \hat{\varepsilon}_{kl}(k')d^3k' + A_{ijkl}(k)\hat{\varepsilon}_{kl}(k) = \delta N_{ijkl}(k)\langle \sigma_{kl} \rangle
\]  

(25)

where one writes

\[
\delta K_{ijkl}(k;k') = e_{irp}e_{jrq}e_{kmn}e_{lon}k_ik_rk_mk_n\delta \hat{C}_{pqmn}(k - k')
\]

\[
\delta N_{ijkl}(k) = e_{irp}e_{jrq}k_r\delta \hat{C}_{pqkl}(k)
\]

\[
A_{ijkl}(k) = e_{irp}e_{jrq}e_{kmn}e_{lon}k_ik_ik_mk_n\langle C_{pqmn} \rangle
\]  

(26)
In deriving (25), use has been made of the fact that the product of \( \delta(k) \) by \( k \) is zero in a
distributional sense. This fact can also be used to express (19) in the alternative form

\[
C_{ijkl} = C_{ijkl}^* - \frac{1}{V} \int \delta N_{mnjl}^*(k) H_{nmpq}(k; k') \delta N_{pqkl}(k') dk d^3k'
\]  

(27)

which shows how the effective moduli trivially reduce to the average moduli for a
homogeneous medium.

Equation (25) is a Fredholm integral equation of the second kind. Successive approximations
to the solution can be obtained as follows. The first-order term of the approximating
sequence is set to

\[
\chi_{i1}^{(1)}(k) = A_{ijkl}^{-1} \delta N_{ijkl}(k) \langle \sigma_{il} \rangle
\]  

(28)

Subsequent terms are defined by means of the following recurrence relation

\[
A_{ijkl}(k) \chi_{i1}^{(n)}(k) = \delta N_{ijkl}(k) \langle \sigma_{il} \rangle - \frac{1}{(2\pi)^3} \int \delta K_{ijkl}(k; k') \chi_{i1}^{(n-1)}(k') dk d^3k'
\]  

(29)

for \( n \geq 2 \). Applying this relation iteratively \( n \) times the \( n \)-th-order approximation to the
stress potential can be given as

\[
\chi_{ij}^{(n)}(k) = \left[ (2\pi)^3 \int H_{ijkl}^{(n)}(k; k') \delta N_{klmn}(k') dk d^3k' \right] \langle \sigma_{mn} \rangle
\]  

(30)

where

\[
H^{(n)} = \frac{1}{(2\pi)^3} \left( I + \sum_{m=1}^{n} \left( -A^{-1} \delta K/2\pi^3 \right)^m \right) A^{-1}
\]  

(31)

is an \( n \)-th-order approximation to the inverse kernel \( H \). These approximate inverse kernels
can be used to formulate an approximating sequence for the effective moduli by writing

\[
C_{ijkl} = C_{ijkl}^* - \frac{1}{V} \int \delta N_{mnjl}^*(k) H_{nmpq}^{(n)}(k; k') \delta N_{pqkl}(k') dk d^3k'
\]  

(32)

This expression results from replacing \( H \) by \( H^{(n)} \) in (19).

It follows from general results concerning Neumann series [20, p. 30] that the
approximating sequences (29) and (32) converge to the exact values of the fluctuation stress
field and effective moduli provided the magnitude of the spatial fluctuations of the elastic
properties is small compared to their average value. In mathematical terms this is expressed
by the requirement that

\[
\| \delta K \| < 1/\| A^{-1} \|
\]  

(33)

where the \( L_2 \)-norm of \( \delta K \) is given by (15) with \( K \) replaced by \( \delta K \) and \( \| A^{-1} \| \) signifies the
euclidean norm of matrix \( A^{-1} \).

4. A TWO-DIMENSIONAL EXAMPLE

The significance of the approximation procedure discussed above is illustrated next by
means of a two-dimensional example. For simplicity, attention is confined to the isotropic
case. Under this assumption, the spatial distribution of elastic properties can be defined
in terms of the form

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in terms of Young's modulus $E$ and Poisson's ratio $\nu$ and the elastic compliances are of the form

$$C_{ab;\rho}(\mathbf{x}) = \frac{1}{2} \left( f(\mathbf{x}) + g(\mathbf{x}) \right) (\delta_{a\eta} \delta_{b\rho} + \delta_{a\rho} \delta_{b\eta}) - g(\mathbf{x}) \delta_{a\rho} \delta_{b\eta}$$  \hspace{1cm} (34)

where greek indices range from 1 to 2 and the scalar functions $f(\mathbf{x})$ and $g(\mathbf{x})$ are defined to be $f = 1/E$, $g = \nu/E$ for plane stress, and $f = (1 - \nu^2)/E$, $g = \nu(1 + \nu)/E$ for plane strain.

The general expressions for the fluctuation stress field and the effective moduli simplify considerably in two dimensions. Thus, the only nonzero component of the stress potential is $\chi_{33}$, which is henceforth denoted by $\chi$. Integral equation (25) reduces to

$$\frac{1}{(2\pi)^2} \int K(k, k') \mathbf{\delta}(k') d^2k' + \langle f \rangle k^2 + \chi_{33} = \delta N_{ab}(k) \langle \sigma_{ab} \rangle$$  \hspace{1cm} (35)

where one writes

$$\delta K(k, k') = A(k, k') \delta f(k - k') - B(k, k') \delta g(k - k')$$

$$A(k, k') = (k_1 k_2' + k_2 k_1')^2, \quad B(k, k') = (k_1 k_2' - k_2 k_1')^2$$

$$\delta N_{ab}(k) = \delta f(k)\delta_{ab} - \delta g(k) k_a k_b$$  \hspace{1cm} (36)

Here, $\langle f \rangle$ and $\langle g \rangle$ are the average values of $f$ and $g$. $\delta f$ and $\delta g$ denote the corresponding spatial fluctuations and $k^2 = k_1^2 + k_2^2$. The first-order terms in the Neumann series expansion are given by

$$\chi^{(1)}(k) = \frac{N_{ab}(k) \langle \sigma_{ab} \rangle}{\langle f \rangle k^4}$$  \hspace{1cm} (37)

for the fluctuation stress potential and by

$$\bar{C}_{ab;\rho}(\mathbf{k}) = \langle C_{ab;\rho}(\mathbf{k}) \rangle - \frac{1}{V} \left\{ \frac{1}{(2\pi)^2} \int \frac{\delta N_{ab}^*(k) \delta N_{ab}(k)}{\langle f \rangle k^4} d^2k \right\}$$  \hspace{1cm} (38)

for the effective moduli. Substituting definition (36c) into eqn (38) one obtains

$$\bar{C}_{ab;\rho}(\mathbf{k}) = \langle C_{ab;\rho}(\mathbf{k}) \rangle = -\frac{1}{2\pi^2 \langle f \rangle} \left\{ \left( \delta_{a\rho} \delta_{b\rho} - \delta_{a\rho} \xi_{a\rho} \xi_{b\rho} - \xi_{a\rho} \delta_{b\rho} + \xi_{a\rho} \xi_{b\rho} \delta_{a\rho} \right) P_{ff}(k) \right.$$

$$+ \left( \delta_{a\rho} \xi_{a\rho} \xi_{b\rho} - \xi_{a\rho} \xi_{b\rho} \delta_{a\rho} + 2 \xi_{a\rho} \xi_{b\rho} \xi_{a\rho} \right) P_{fg}(k)$$

$$+ \xi_{a\rho} \xi_{b\rho} \xi_{a\rho} \xi_{b\rho} P_{gg}(k) \right\} d^2k$$

In this expression the symbol $\xi$ is used to signify the unit position vector $\mathbf{k}/k$ in Fourier space, and $P_{ff}(k)$, $P_{fg}(k)$ and $P_{gg}(k)$ are the power spectra of $\delta f$ and $\delta g$ and the cross-power spectrum of $\delta f$ and $\delta g$, respectively, i.e.

$$P_{ff}(k) = \frac{1}{V} \delta f^* \delta f, \quad P_{gg}(k) = \frac{1}{V} \delta g^* \delta g$$

$$P_{fg}(k) = \frac{1}{V} \delta f^* \delta g = \frac{1}{V} \delta g^* \delta f$$  \hspace{1cm} (40)
The power spectra are the Fourier transforms of the spatial correlation functions

\[ R_{ff}(r) = \langle \delta f(x)\delta f(x + r) \rangle \quad R_{gg}(r) = \langle \delta g(x)\delta g(x + r) \rangle \]

\[ R_{fg}(r) = \langle \delta f(x)\delta g(x + r) \rangle = \langle \delta g(x)\delta f(x + r) \rangle \]  

(41)
i.e. \( P_{ff} = \hat{R}_{ff} \), \( P_{gg} = \hat{R}_{gg} \) and \( P_{fg} = \hat{R}_{fg} \). Note that embodied in definitions (41) is the assumption that the material is statistically homogeneous, i.e. the spatial correlations are independent of \( x \).

It is seen from (39) that the first-order approximation to the effective moduli incorporates the information concerning the distribution of elastic properties embodied in the two-point spatial correlation functions. It is readily checked that higher order approximations gradually incorporate the higher order spatial correlations. Thus, in practice the number of terms after which the Neumann series is truncated is dictated not only by accuracy considerations but also by the amount of microstructural information available. For random media, spatial correlations are rarely known beyond the two-point correlation functions. For these cases, the highest approximation which can be obtained is given by the first-order term. This should provide reasonably accurate estimates for mildly heterogeneous media in which the elastic properties are nearly uniform.

Further insight into the nature of (39) can be gained as follows. Introducing polar coordinates \((k, \theta)\) in Fourier space, the unit position vector can be expressed as \( \xi = (\cos\theta, \sin\theta) \), and the differential of volume becomes \( d^2k = kd\kappa d\theta \). If the spatial correlations are isotropic, i.e. depend only on \( k \) and not on \( \theta \), then (39) takes the form

\[
\bar{C}_{\alpha\beta\gamma\delta} - \langle C_{\alpha\beta\gamma\delta} \rangle = -\frac{1}{2\pi} \int_0^{2\pi} \left[ (\delta_{\alpha\beta} \delta_{\gamma\delta} - \delta_{\alpha\beta} \delta_{\gamma\delta} - \xi_{\alpha\beta} \xi_{\gamma\delta} + \xi_{\alpha\beta} \xi_{\gamma\delta}) \langle \delta f \delta f \rangle \\
+ (-\delta_{\alpha\beta} \delta_{\gamma\delta} \xi_{\gamma\delta} - \xi_{\alpha\beta} \delta_{\gamma\delta} + 2\xi_{\alpha\beta} \xi_{\gamma\delta}) \langle \delta f \delta g \rangle \\
+ \xi_{\alpha\beta} \xi_{\gamma\delta} \xi_{\gamma\delta} \langle \delta g \delta g \rangle \right] d\theta
\]

(42)

where

\[
\langle \delta f, \delta f \rangle = R_{ff}(0) = \frac{1}{(2\pi)^2} \int P_{ff}(k) d^2k
\]

\[
\langle \delta f, \delta g \rangle = R_{fg}(0) = \frac{1}{(2\pi)^2} \int P_{fg}(k) d^2k
\]

\[
\langle \delta g, \delta g \rangle = R_{gg}(0) = \frac{1}{(2\pi)^2} \int P_{gg}(k) d^2k
\]

(43)

are the joint variances of \( f \) and \( g \). Thus, in the case of circular symmetry the first-order effective moduli are entirely determined by one-point statistics, namely the average values and joint variances of \( f \) and \( g \). The integrals involved in (42) are readily evaluated yielding

\[
\bar{C}_{\alpha\beta\gamma\delta}(x) = \frac{1}{2} \left( \langle f \rangle + \langle g \rangle \right) (\delta_{\alpha\beta} \delta_{\gamma\delta} + \delta_{\alpha\beta} \delta_{\gamma\delta}) - \langle f \rangle \delta_{\alpha\beta} \delta_{\gamma\delta}
\]

\[
- \frac{1}{(2\pi)^2} \int \left( \frac{1}{6} \langle \delta f \delta f \rangle + 2\langle \delta f \delta g \rangle + \frac{1}{3} \langle \delta f \delta g \rangle \right) (\delta_{\alpha\beta} \delta_{\gamma\delta} + \delta_{\alpha\beta} \delta_{\gamma\delta})
\]

\[
- \frac{1}{(2\pi)^2} \int \left( \frac{1}{8} \langle \delta f \delta f \rangle - \frac{3}{4} \langle \delta f \delta g \rangle + \frac{1}{8} \langle \delta f \delta g \rangle \right) \delta_{\alpha\beta} \delta_{\gamma\delta}
\]

(44)

As can be seen, the overall response is that of an isotropic elastic material.
As a particular example, let us consider the case of a random distribution of inclusions in a homogeneous matrix. We allow the inclusions to have arbitrary shapes but we require that they be distributed with no preferred orientation in space. This assumption together with the randomness in the position of the inclusions results in circularly symmetric correlations and (44) applies. Let $\alpha_1$ and $\alpha_2$ denote the area fractions of matrix and inclusions, and $f_1, g_1, f_2$ and $g_2$ the values of $f$ and $g$ within the matrix and the inclusions, respectively. Then straightforward computations show that

\begin{align*}
\langle f \rangle &= \alpha_1 f_1 + \alpha_2 f_2 \\
\langle g \rangle &= \alpha_1 g_1 + \alpha_2 g_2 \\
\langle \delta f \delta f \rangle &= \alpha_1 \alpha_2 (f_1 - f_2)^2 \\
\langle \delta f \delta g \rangle &= \alpha_1 \alpha_2 (f_1 - f_2)(g_1 - g_2) \\
\langle \delta g \delta g \rangle &= \alpha_1 \alpha_2 (g_1 - g_2)^2
\end{align*}

(45)
Fig. 2. Plane strain effective moduli of a fiber reinforced plastic with $E_1 = 0.4 \times 10^6$, $E_2 = 10.5 \times 10^6$, $\nu_1 = 0.35$ and $\nu_2 = 0.20$.

These relations, together with (44) completely determine the first-order effective moduli. When spatial correlations do not exhibit circular symmetry, eqn (39) takes the form

$$C_{a\beta}^{(1)} - \langle C_{a\beta} \rangle = -\frac{1}{2\pi^2 \langle f \rangle} \int_0^{2\pi} \left[ \left( \delta_{a\beta} \delta_{\gamma\delta} - \delta_{a\gamma} \delta_{\beta\delta} - \xi_{a\beta} \delta_{\gamma\delta} + \xi_{a\gamma} \delta_{\beta\delta} \right) P_{ff}(\theta) 
+ ( - \delta_{a\beta} \xi_{\gamma\delta} - \xi_{a\gamma} \delta_{\beta\delta} + 2 \xi_{a\gamma} \xi_{\beta\delta} ) P_{fg}(\theta) 
+ \xi_{a\gamma} \xi_{\beta\delta} P_{gg}(\theta) \right] d\theta$$

(46)

where the angular correlations are defined to be

$$P_{ff}(\theta) = \int_0^\infty P_{ff}(k)dk 
\quad P_{fg}(\theta) = \int_0^\infty P_{fg}(k)dk 
\quad P_{gg}(\theta) = \int_0^\infty P_{gg}(k)dk$$

(47)

The angular correlation functions indicate how the elastic properties are polarized in different direct anisotropic, uncase circular syp.

Figures 1 and 2 show the bounds for fiber reinforced materials, $E_1 = 10.5 \times 10^6$. Voigt [1, 2] and Hashin-Rosen bounds for uniform stress, respectively. The computed values of $E_1$, $E_2$, and $\nu$ are close to the Hashin-Rosen bounds for a wide range of volume fractions.

The first example is with $\nu_1 = 0.2$. For other cases, the shear moduli are computed and the computed values are close to the Hashin-Rosen bounds for fibers.

A second class of properties of $E_1 = 10.5 \times 10^6$. disimilar, a set of order perturbations reasonable values of $\nu_1$ et formula in en.
different directions. It is interesting to note that the effective moduli (46) are in general anisotropic, unless the angular correlations (47) are uniform over the unit circle in which case circular symmetry is obtained.

5. NUMERICAL EXAMPLES

Figures 1 and 2 show a comparison between the present method, the Hashin–Rosen bounds for fiber reinforced composites [21], and the estimates obtained by assuming uniform stress or uniform strain throughout the material. This latter method goes back to Voigt [1, 2] and Taylor [4, 5], and is also known as the law of mixtures. The assumptions of uniform stress or strain provide coarse lower and upper bounds on the effective behavior, respectively. The Hashin–Rosen bounds are always more restrictive and in the case of the bulk modulus the lower and upper bounds coincide, Figs 1a, 2a. In fact, the bounds provide an exact result for a composite with a continuous gradation of inclusions such that all subvolumes of the composite contain both phases in fixed proportions [21]. The Hashin–Rosen bounds for the shear modulus do not in general coincide and steadily deteriorate for phases of increasing dissimilarity.

The first example concerns a composite material in which $E_2 = 2E_1$, $\nu_1 = 0.3$ and $\nu_2 = 0.2$. Figure 1a shows the plane strain bulk modulus $k = \lambda + \mu$ and Fig. 1b depicts the shear modulus. As expected, the coarse bounds contain the Hashin–Rosen bounds in all cases. The bulk moduli computed from eqns (44) and (45) are seen to replicate very closely the Hashin–Rosen exact result over the whole range of volume fractions. The computed shear moduli fall within the Hashin–Rosen bounds, which in this case are remarkably tight.

A second example is concerned with a class of glass fiber-reinforced plastics. The properties of the binder are $E_1 = 0.4 \times 10^5$, $\nu_1 = 0.35$, whereas those of the fibers are $E_2 = 10.5 \times 10^6$, $\nu_2 = 0.2$. It should be emphasized that in this case the phases are vastly dissimilar, a situation which would appear to be outside the range of applicability of a first-order perturbation result. In spite of these demanding conditions the method yields reasonable values for low enough concentrations of fibers. Thus, beyond $\varepsilon_1 = 0.4$ the computed bulk modulus is in good agreement with the Hashin–Rosen exact result, Fig. 2a. The computed shear modulus matches closely the Hashin–Rosen lower bound for values of $\varepsilon_1$ within the same range. Such lower bound is frequently taken as a design formula in engineering applications [11].

**REFERENCES**

APPENDIX A

Some basic properties of the Fourier transform are recorded next. The Fourier transform of a function \( f(x) \) of the spatial coordinates \( x \equiv (x_1, x_2, x_3) \) is defined as

\[
\hat{f}(k) = \left( \frac{1}{2\pi} \right)^{3/2} \int f(x) e^{-ix \cdot k} \, dx
\]  
(A1)

For instance, if \( f(x) = c \equiv \text{const.} \) one has

\[
\hat{f}(k) = (2\pi)^{3/2} c \delta(k)
\]  
(A2)

where \( \delta(k) \) signifies the Dirac delta distribution at the origin.

The following standard results are used throughout the paper

\[
\langle \hat{f}\hat{g} \rangle = \lim_{R \to \infty} \frac{1}{V} \int_{V} \hat{f}(k) \hat{g}(k) e^{i\mathbf{k} \cdot \mathbf{r}} \, d^3k
\]

\[
\hat{f}(k) = \left( \frac{1}{2\pi} \right)^{3/2} \int f(k - k') \delta(k') \, d^3k'
\]  
(A3)

In the first identity \( \hat{f} \) and \( \hat{g} \) signify the restrictions of \( f \) and \( g \) to a closed compact subset \( V \) of \( R^3 \) containing the origin. In terms of the limiting process indicated in (A3a), \( \hat{f} \) and \( \hat{g} \) are to be regarded as members of two nets of functions obtained by letting the domain \( V \) cover the whole space. For example \( V \) can be taken to be the ball of radius \( R \) centered at the origin and the limit in (A3) reduces to letting \( R \to \infty \). It is implicitly assumed that due to statistical disorder the result is independent of the choice of origin. For simplicity of notation, the subindex \( V \) in the integrand of (A3) and the limit sign are dropped throughout the paper.

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