AN ANALYTICAL STUDY OF THE LOCALIZED FAILURE MODES OF CONCRETE

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A theoretical framework for the analysis of localized failure in concrete is presented. The theory is predicated upon the assumption that discrete failure planes arise as a result of a process of localization of damage. The onset of localized modes is characterized as a bifurcation phenomenon whereby local neighborhoods of the material depart from near-uniform straining in favor of highly localized deformation patterns. Simple bifurcation techniques are discussed which suffice to detect when localization initiates and to determine the geometry of the localized deformation modes. Localization techniques are seen to provide a simple yet effective means of extending the range of applicability of traditional distributed damage models to situations of localized failure. Numerical calculations for biaxial stress paths exhibit a good overall agreement with experimental observations.

1. Introduction

The inelastic behavior of a broad class of materials of engineering interest can be traced to two strongly interacting mechanisms:
- nucleation and growth of microcracks, and
- plastic flow.

Such materials include rocks and concrete as well as ceramics and ceramic matrix composites, and have been frequently referred to as plastic-fracturing materials. The proper characterization of plastic-fracturing behavior is presently the subject of considerable conjecture. For these materials, some fundamental questions have been raised which relate to the very foundations of continuum theories. In particular, whether or not the process of damage undergone by plastic-fracturing materials can be treated as a continuously distributed phenomenon is presently the subject of an ongoing controversy. The available experimental evidence lends support to advocates of both continuous and discrete theories. Thus, acoustic measurements and crack surveys show that microcrack growth is a fairly distributed process during early stages of loading. By contrast, for certain stress paths and beyond certain critical conditions localized damage prevails. Subsequent damage is confined to narrow bands which eventually collapse into discrete failure planes. On the basis of this latter mechanism, it has been argued that constitutive theories based on a continuum hypothesis are not representative of the observed phenomena. On the other hand, proponents of the continuous picture invoke averaging or smearing formalisms whereby continuum mechanical theories can be brought to bear on situations of highly localized damage.

Here, an alternative hypothesis is investigated, namely, that localization of damage can be understood as an instability in the inelastic behavior of the material. An explanation of this kind has been proposed by Rudnicki and Rice (1975) for the inception of faulting in brittle rock masses under compression. This paper aims at assessing the ability of localization techniques to reproduce experimental observations concerning the failure of concrete. The conditions under which localized deformation modes arise are currently well-understood (see e.g., Rice, 1976). However, in the past preferential attention has been given to the study of shear band formation in metals and soils, and the possibility of applying localization techniques to plastic-fracturing materials has remained largely unexplored. The present analysis is based on a constitutive model proposed by Ortiz (1985) and
summarized in Section 3. The predictions of the localization analysis are given in Section 4. It is observed that the localized failure modes of concrete range from pure splitting modes to predominantly shear type failure. Furthermore, under certain stress paths localized modes do not arise and failure takes place as a consequence of distributed damage. Thus, even in materials where the possibility of localized damage arises, models of distributed damage can play a useful role for the following reasons:

- Damage takes place in a distributed fashion prior to the onset of localization.
- For certain stress paths, localized modes do not arise and damage remains of a distributed type up to failure.
- In conjunction with localization techniques, models of distributed damage are of potential value as a basis for the study of the onset of localization and subsequent development of localized failure modes.

From this perspective, localization techniques appear as a complement to traditional constitutive models, and a convenient mathematical formalism whereby localized damage and failure can be accounted for.

2. Localization of inelastic deformations

In this section we review some aspects of the general theory of localization of inelastic deformations. Some of the basic principles underlying the theory follow from Hadamard’s studies of elastic stability (Hadamard, 1903), extended to the inelastic context by Thomas (1961), Hill (1962) and Mandel (1965). A thorough discussion of the subject can be found in review article of Rice (1976). Here, for simplicity attention is confined to infinitesimal deformations and thermally decoupled, rate-independent behavior.

2.1. Localization condition

Consider a homogeneous, homogeneously deformed solid subjected to quasi-static increments of deformation $\varepsilon$. We wish to determine if a bifurcation can occur in such a manner that subsequent deformations become discontinuous across a plane of orientation $n$. Let $u$ be the displacement field in the solid. Whereas $u$ itself remains continuous after the onset of localization, the displacement gradients $\nabla u$ will exhibit a jump across the plane of discontinuity, i.e.,

$$[u_{ij}] = u_{ij}^+ - u_{ij}^- \neq 0$$

where the superindex $+$ refers to the plus side of the plane of discontinuity and $-$ to the minus side. Maxwell's compatibility conditions necessitate that the jump (2.1) be of the form

$$[u_{ij}] = g_{ij}n_j$$

for some vector $g$. Let us further define $m$ to be the unit vector along $g$, i.e.,

$$m_i = g_i/g, \quad g = |g|.$$  \hfill (2.3)

The pair of unit vectors $m$ and $n$ entirely define the nature of the discontinuity. In particular, the corresponding strain jump is

$$[\varepsilon_{ij}] = \frac{1}{2}(g_{ij} + g_{ij}n_in_i).$$  \hfill (2.4)

Often, two parallel planes of discontinuity pair up to form a band. Then, the displacement field corresponding to the localized mode is confined to the region within the band.

Two limiting cases are noteworthy:

(a) $m$ orthogonal to $n$. The material in the band deforms in pure shear, i.e., a shear band develops.

(b) $m$ parallel to $n$. The band undergoes extension normal to the planes of discontinuity and a splitting failure mode results. We shall refer to these types of bands as separation bands. Shear bands are characteristic of materials exhibiting isochoric plasticity. On the other hand, splitting failure modes can be thought of as idealizations of the separation processes which occur during progressive brittle failure. In between these two extremes lies a continuous spectrum of mixed failure modes for which $m$ and $n$ are neither orthogonal nor parallel. The angle formed by $m$ and $n$

$$\psi = \arccos(m,n)$$  \hfill (2.5)

provides an indication of the type of failure. Thus, the case $\psi = 0^\circ$ corresponds to a separation band...
and $\psi = 90^\circ$ to a shear band. Failure modes with values of $\psi$ in the range $(0, 90)$ exhibit both separation and shear plastic flow taking place concurrently. Plastic-fracturing materials such as concrete are capable of failure in mixed modes with angles $\psi$ lying anywhere between $0^\circ$ and $90^\circ$. Pure splitting and shear failure modes are not precluded either. Experimental evidence pointing to this duality of failure modes in concrete is extensive and dates back to the origins of experimental concrete research (see e.g., Bellamy, 1961; Rosenthal and Glucklich, 1970; Mills and Zimmerman, 1970).

Next we investigate under what conditions localized failure modes are possible. To this end, let us assume that the solid is at the onset of localization. Since the localized mode has not built up yet to any appreciable extent, stresses $\sigma$ and strains $\varepsilon$ are still continuous throughout the body but not so stress rates $\dot{\sigma}$ and strain rates $\dot{\varepsilon}$. Assuming rate-independent behavior, the incremental stress-strain relations take the form

$$\dot{\sigma}_{ij} = D_{ijkl}^{(T)} \dot{\varepsilon}_{kl}$$

(2.6)

where $D^{(T)}$ is the tangent stiffness of the material. To obtain the lowest bifurcation point, the same plastic loading branch of the moduli $D^{(T)}$ is assumed operative on both sides of the incipient plane of discontinuity. This corresponds to investigating bifurcation for a linear comparison solid (Hill, 1958; Raniecki and Bruhns, 1981). Taking jumps in (2.6) leads to

$$\left[ \dot{\sigma}_{ij} \right] = D_{ijkl}^{(T)} \left[ \dot{\varepsilon}_{kl} \right].$$

(2.7)

Equilibrium across the discontinuity planes requires that the tractions $t$ be continuous, i.e.,

$$\left[ t_j \right] = \left[ n_i \dot{\sigma}_{ij} \right] = n_i \left[ \dot{\sigma}_{ij} \right] = 0.$$  

(2.8)

Combining (2.7) and (2.8), it follows that

$$n_i D_{ijkl}^{(T)} \left[ \dot{\varepsilon}_{kl} \right] = 0.$$  

(2.9)

Finally, bringing in the kinematic relation (2.4) and definition (2.3) we obtain

$$(n_i D_{ijkl}^{(T)} n_i) m_k = 0.$$  

(2.10)

This condition has to be satisfied by $m$ and $n$ for the localized mode to be possible. The onset of localization occurs at the first point in the deformation history for which a nontrivial solution of (2.10) exists. Introducing the notation

$$A_{jk}(n) \equiv n_i D_{ijkl}^{(T)} n_i$$

(2.11)

the localization condition becomes

$$A_{jk}(n) m_k = 0.$$  

(2.12)

Thus, for localization to occur along the direction $n$, the localization matrix $A(n)$ has to have at least one zero eigenvalue. This in turn necessitates

$$f(n) = \det(A(n)) = 0$$  

(2.13)

which constitutes an alternative form of the localization condition (2.10). If a unit vector $n$ satisfying (2.13) can be found, the corresponding vector $m$ which completes the definition of the localized mode follows from (2.12).

2.2. Computation of localization directions $m$ and $n$

Next we discuss a computational procedure for detecting the onset of localization and determining the localization directions $m$ and $n$. If the material response is path-dependent, the integration of the constitutive equations has to be carried out incrementally. If, for instance, the instantaneous initial response of the material is isotropic-elastic, the localization function $f$ takes initially the value

$$f(n) = (\lambda_0 + 2\mu_0) \mu_0^2$$  

(2.14)

where $\lambda_0$ and $\mu_0$ are the Lamé constants of the virgin material. Note that $f(n) > 0$ and independent of $n$. As the process of deformation progresses, $f$ may develop minima which eventually become negative or zero. This in turn signals the onset of localization. To detect precisely when this happens, the minima of the localization function $f$ can be computed at every deformation increment and the localization condition (2.13) checked at the minima. This leads to considering the constrained minimization problem

minimize $f(n) = \det(n_i D_{ijkl}^{(T)} n_i)$

subject to $|n| = 1$.
where $D^{(T)}$ is the current value of the tangent moduli. The minima are characterized by the condition

$$
\frac{\partial}{\partial n_i} \left[ f(n) - \lambda |n|^2 \right] = \frac{\partial f(n)}{\partial n_i} - 2\lambda n_i = 0
$$

(2.16)

where $\lambda$ is a Lagrange multiplier. Differentiating $f$ in (2.16) one obtains

$$
\text{det}(A(n)) D^{'(T)}_{ijkl} A^{-1}_{kl}(n) n_i - \lambda n_i = 0.
$$

(2.17)

Introducing the notation

$$
J_{\mu}(n) = \text{det}(A(n)) D^{'(T)}_{ijkl} A^{-1}_{kl}(n)
$$

(2.18)

the minimum condition (2.17) can be recast as

$$
J_{\mu}(n) n_i - \lambda n_i = 0.
$$

(2.19)

The solutions of (2.19) can be found in two steps:

(i) Expressing $n$ in terms of spherical angles, i.e., setting $n = (\cos \phi \cos \theta, \cos \phi \sin \theta, \sin \phi)$, the range of variation $[0, \pi] \times [0, \frac{\pi}{2}]$ of $(\theta, \phi)$ is swept at 5-degree increments to determine a first approximation $n^{(0)}$ to the minima.

(ii) The locations of the minima are then pinpointed by means of the iterative scheme:

$$
J_{\mu}(n^{(k)}) n^{(k+1)} - \lambda^{(k+1)} n^{(k+1)} = 0.
$$

(2.20)

Thus, at every iteration an eigenvalue problem is formulated based on the matrix $J$ evaluated at the previous iteration $n^{(k)}$. The minimum eigenvector of (2.20) is taken as the new iterate $n^{(k+1)}$.

Once the orientations of the discontinuity planes have been determined, the corresponding $m$-vectors are computed as the zero-eigenvectors of the localization matrix $A(n)$.

3. A constitutive model for concrete

Past research has shown that the localization criteria discussed in Section 2 are quite sensitive to subtle features of the constitutive model such as the curvature of the yield surface (Mear and Hutchinson, 1985) and yield surface vertex effects (Hill and Hutchinson, 1975; Rudnicki and Rice, 1975), deviations from normality (Rice, 1976; Rudnicki and Rice, 1975), dilatational plastic flow (Needleman and Rice, 1978; Rice, 1976), and rate dependence (Needleman and Rice, 1978). This illustrates the fact that the success of localization techniques in characterizing failure depends critically on having at hand realistic descriptions of the inelastic behavior. This section summarizes some relevant features of a general theory of inelasticity and failure of concrete proposed by Ortiz (1985). The theory aims at formulating a general framework for the constitutive modeling of concrete based on first principles of mechanics. The main building blocks of the formulation are:

- The theory of interacting continua or mixture theory.
- A rate-independent theory of damage.

The importance of imperfections and heterogeneity in the development of localized failure modes has been noted elsewhere (Rice, 1976). Mixture theory is a particularly convenient way of taking into account the strong heterogeneity of concrete. To this end, concrete can be idealized as a mixture comprising two main phases: mortar and aggregate. A far-reaching consequence of this fact is that the externally applied stresses distribute unequally between the two phases. The average stresses acting in mortar and aggregate must jointly equilibrate the applied loads but may be vastly different from each other. It seems reasonable to assume that these phase stresses drive the inelastic mechanisms of damage in mortar and plastic flow in aggregate. In other words, the inelastic mechanisms are driven by stresses which may differ substantially from the applied ones. Mixture theories provide a simple yet effective means of estimating the value of the phase stresses that develop in mortar and aggregate. On the basis of this model, it has been argued by Ortiz (1985) that the composite nature of concrete lies at the foundation of certain aspects of the observed phenomenology of concrete, some of which had heretofore defied satisfactory explanation. Thus, mixture theories predict that purely compressive uniaxial loads induce large tensile stresses in mortar which act normal to the axis of loading. The development of these tensile stresses explains the observed splitting failure modes under compressive loading. Another aspect of the behavior of concrete which is amenable to a plausible ex-
planation within the confines of mixture theory is the hysteretic unloading loops that arise under cyclic compression. A brief overview of mixture theory as it bears on concrete is given in section 3.1.

Other authors have proposed alternative explanations for the axial splitting effect. For instance, Nemat-Nasser (1985) conducted experiments on idealized planar models involving hard inclusions in a brittle matrix. The experiments suggest that the cracks nucleate at points on the interface and propagate into the matrix parallel to the axis of compressive loading. However, a precise constitutive model has not yet emerged which bridges the gap between these observations and the overall behavior of a material such as concrete. Another micromechanical theory which aims at explaining splitting in compression is the so-called sliding crack model (see e.g., Nemat-Nasser and Hori, 1982; Hori and Nemat-Nasser, 1985; 1986). In spite of the theoretical appeal of the theory, most scanning electron microscope observations on stressed rock samples indicate that a high proportion of the stress induced microcracks open up in mode I (Wong, 1985, p. 272). This has spurred a number of studies (Holcomb and Stevens, 1980; Janach and Guex, 1980; Dey and Wang, 1981; Costin, 1983) which critically appraise the relevance of the sliding crack model and develop alternatives not involving frictional slip. By contrast, there are also recent data based on acoustic emission measurement which are indicative of shear motion (Sondergeld and Estey, 1982; Yanagidani et al., 1985). A comparison of theoretical predictions and quantitative SEM data carried out by Wong (1985) shows that the sliding crack model is not in disagreement with the experimental observations. Thus, it seems that a consensus concerning the micromechanical foundations of compressive damage is far from having been reached. Whatever the micromechanical picture adopted, the constitutive framework discussed below would appear to be general enough to accommodate in an overall sense a wide variety of micromechanical viewpoints.

Once the relation between the behavior of the phases and that of the mixture has been established, the task that remains is to characterize the constitutive response of the phases as separate materials. On one hand, aggregate is a granular medium and as such can be adequately characterized within the bounds of classical plasticity, see section 3.2. On the other hand, the main inelastic mechanism underlying the behavior of mortar is damage. The problem of characterizing the process of elastic degradation of mortar is compounded by the following observations:

- damage results in strong elastic anisotropy,
- closure of microcracks under load reversal results in a sudden stiffening of the material,
- damage progresses both under tension and compression,
- microcrack growth is not perfectly brittle but involves plastic deformations,

A theory of damage which allows for general elastic anisotropy can be constructed by taking the flexibility compliances themselves as the measures of damage (Ortiz, 1985). A principal objective of the theory is then to characterize the evolution in time of the elastic compliances. Microcrack closure can be formulated as a unilateral constraint whereby the added deformations due to active microcracks must be tensile in all directions. This unilateral constraint determines which fraction of microcracks is opening and which one is closing under a given state of stress. As a result, the current value of the elastic compliances depends not only on the extent of damage undergone by the material but also on the state of stress which determines the active microcrack fraction. When compared with experimental data, the model is seen to adequately capture the main features of the behavior of concrete under uniaxial monotonic tension and compression, cyclic uniaxial compression, biaxial compression and tension and triaxial conditions. A simplified version of the theory is presented in section 3.3.

3.1. Concrete as a mixture

Concrete may be idealized as a mixture comprising two main phases, mortar and aggregate (Ortiz and Popov, 1982). Built into the notion of 'mixture' is the assumption that an arbitrarily small volume of concrete contains both mortar and aggregate in fixed volumetric fractions, \( \alpha \).
and $\alpha_2$. This has the effect of eliminating from the formulation all microstructural length scales such as the aggregate size. However, 'size effects' are known to influence the behavior of concrete samples whose characteristic dimensions are comparable to the aggregate size (Blanks and McNamara, 1935; Neville, 1966; Ravina, 1973; Walker and Bloem, 1960). As the dimensions of the sample are increased to about twenty times the aggregate size, it is observed that mechanical properties such as strength stabilize about a fixed, size independent value (Blanks and McNamara, 1935) and the influence of the microscale is lost. Therefore, on the basis of this experimental evidence it may be concluded that mixture theories are applicable when the processes of interest have characteristic lengths which are large compared to the size of the aggregate.

Following Green and Naghdi (1965), the conservation laws for a mixture may be obtained by requiring certain invariance properties from an energy balance equation. The case of concrete is particularly simple due to the absence of diffusion between the phases. Let $\sigma_1$ and $\sigma_2$ denote the average or phase stresses acting in mortar and aggregate, respectively, and $\sigma$ the applied stresses. The requirement that $\sigma_1$ and $\sigma_2$ jointly equilibrate $\sigma$ can be expressed

$$\sigma = \alpha_1 \sigma_1 + \alpha_2 \sigma_2.$$  \hspace{1cm} (3.1)

It should be emphasized that the phase stresses $\sigma_1$ and $\sigma_2$ have to be understood in the sense of the theory of interacting continua. Thus, the phase stresses are macroscopic variables pertaining to material neighborhoods which are large compared to the microstructure. In particular, $\sigma_2$ is not meant to represent the state of stress within the aggregate particles but to provide an average measure of the interaction forces that develop between them (for a micromechanical definition of stress in granular media see, e.g., Christoffersen et al., 1981; Oda et al., 1980). It is interesting to note that although angular momentum balance necessitates that $\sigma$ be symmetric, the phase stresses $\sigma_1$ and $\sigma_2$ may be non-symmetric. In fact, in many types of mixtures the phases do interact through stress couples, which results in non-symmetric phase stresses. For the discussion that follows this effect is neglected.

The absence of diffusion between the phases necessitates compatibility of macroscopic deformations, i.e.,

$$\epsilon_1 = \epsilon_2 = \epsilon$$ \hspace{1cm} (3.2)

where $\epsilon_1$, $\epsilon_2$ and $\epsilon$ signify the macroscopic strain tensors of mortar, aggregate and concrete, respectively. As in the case of the phase stresses, it should be strongly emphasized that these strain tensors refer to the macroscopic deformation of the phases and concrete and are not intended as measures of the deformation processes that take place at the microscopic level. In particular, $\epsilon_2$ does not pertain to the state of deformation within the aggregate particles but it rather describes the overall deformation undergone by volumes of aggregate containing many particles. Thus, the compatibility relation (3.2) takes a completely different meaning than the similar one upon which Taylor's method for composite materials is predicated (Taylor, 1938). In this latter case, compatibility of deformations between the matrix and the inclusions is postulated at the microstructural level. This is a rather strong assumption which has been relaxed in various ways in subsequent refinements of the theory (see e.g., Mura, 1982, for a review).

In the mixture model used here, compatibility of deformations is enforced at the macroscopic level and is a direct consequence of the absence of diffusion between the phases.

Composition rule (3.1) and compatibility condition (3.2) are all the relations from mixture theory that are needed to derive the overall stress-strain relations for concrete from those of its constituents. To illustrate this point, let us assume that the stress-strain relations of both mortar and aggregate are of the general form

$$\epsilon_1 = C_1 : \sigma_1 + \epsilon_1^p,$$

$$\epsilon_2 = C_2 : \sigma_2 + \epsilon_2^p$$ \hspace{1cm} (3.3)

where $C_1$ and $C_2$ are the current flexibility compliances of mortar and aggregate and $\epsilon_1^p$ and $\epsilon_2^p$ the corresponding plastic deformations. The symbol $(;)$ in (3.3) is used to signify doubly contracted product, e.g., $(C; \sigma)_{ij} = C_{ijk} \sigma_{kj}$. Making

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use of the mixture-theoretical relations (3.1), (3.2),
it is found that (Ortiz and Popov, 1982; Ortiz,
1986, 1987)

\[ \sigma_1 = B_1 : \sigma + \rho_1, \]
\[ \sigma_2 = B_2 : \sigma + \rho_2, \]  
(3.4)

where the influence tensors \( B_1 \) and \( B_2 \) and the
residual stresses \( \rho_1 \) and \( \rho_2 \) are defined as

\[ B_1 = \frac{D_1}{C}, \quad \rho_1 = \frac{D_1}{C} : (\epsilon^p - \epsilon^1), \]
\[ B_2 = \frac{D_1}{C}, \quad \rho_2 = \frac{D_2}{C} : (\epsilon^p - \epsilon^2). \]  
(3.5)

Here, \( D_1 = C_1^{-1} \) and \( D_2 = C_2^{-1} \) are the phase stiffness compliances and one writes

\[ D = \alpha_1 D_1 + \alpha_2 D_2, \quad C = D^{-1}, \]
\[ \epsilon^p = C : (\alpha_1 D_1 : \epsilon^p + \alpha_2 D_2 : \epsilon^p) \]  
(3.6)

for the stiffness tensor and plastic deformation of concrete.

Some aspects of (3.4) merit further comment. Thus, it is interesting to note that the phase stresses are the sum of two terms: a load induced term and residual stresses. The former arises as a direct consequence of the stresses \( \sigma \) that are applied to concrete. The influence tensors determine how these stresses distribute between mortar and aggregate. The value of the influence tensors is in turn determined by the relative stiffness of mortar, aggregate and concrete. The residual stresses are those that remain in the phase upon removal of the external loads. From their definition, it is apparent that the residual stresses are the result of the plastic flow of the phases.

It was noted by Ortiz (1985) that the unloading hysteretic loops that arise during cyclic uniaxial compression can be attributed to the development of residual stresses \( \rho_2 \) in aggregate. It was also noted by Ortiz and Popov (1982) and by Ortiz (1986) that mixture theories such as the one outlined above predict the development of significant tensile stresses in mortar under purely compressive loads. Thus, in uniaxial compression the theory predicts tensile stresses acting in mortar normal to the axis of loading. Under biaxial compression, the splitting stresses are predicted to occur normal to the plane of loading. By contrast, under uniaxial tension the tensile stresses that appear in mortar are aligned with the applied loads. Thus, it is seen that the predicted orientation of the tensile induced in mortar is closely related to the orientation of the failure modes that are observed experimentally. However, as discussed in Section 4, the planes of failure are not necessarily normal to the largest tensile principal stress. The deviation is due to effect of aggregate which fails primarily in shear. Thus, in general, the overall failure modes of concrete are a combination of a shear and a separation band. This situation is to be expected from the fact that concrete is a mixture of a plastic (aggregate) and a fracturing (mortar) material.

In section 3.2 and 3.3 two material models are proposed for mortar and aggregate which together with the equations of mixture theory provide a complete description of the constitutive behavior of concrete. The models proposed for mortar and aggregate are both rate-independent. Following a standard procedure it is possible to formulate incremental stress-strain relations of the type

\[ \dot{\sigma}_1 = D_1^{(T)} : \dot{\epsilon}_1, \quad \dot{\sigma}_2 = D_2^{(T)} : \dot{\epsilon}_2 \]  
(3.7)

where \( D_1^{(T)} \) and \( D_2^{(T)} \) denote the tangent stiffness compliances of mortar and aggregate, respectively. Making use of (3.1) and (3.2) one finds

\[ \dot{\sigma} = D^{(T)} : \dot{\epsilon} \]  
(3.8)

where

\[ D^{(T)} = \alpha_1 D_1^{(T)} + \alpha_2 D_2^{(T)} \]  
(3.9)

are the tangent stiffness compliances of concrete. These provide the necessary input for the localization techniques discussed in Section 2.

3.2. A damage model for mortar

In this section, a simplified version of a damage model for mortar proposed by Ortiz (1985) is briefly outlined. For simplicity of notation, the index 1 identifying mortar as one of the phases of concrete is dropped throughout this section. The model allows for general induced anisotropy, damage in tension and compression, unstable postfailure behavior, and incorporates the effect of microcrack closure. This latter effect arises during load reversals and may result in a sudden stiffening of the material. Microcrack closure can
be built into the formulation as a unilateral constraint which requires that the deformation contributed by microcracks opening in tension be one of stretching in all directions. However, in the present work, the primary emphasis is on processes of loading that are proportional and monotonic up to failure. Under these conditions, microcrack closure is of no concern and the formulation of the model simplifies considerably. In the discussion that follows these simplifying assumptions are made.

A distinct characteristic of concrete is that its elastic properties degrade as a consequence of microcrack extension. Crack surveys show that under general loading conditions crack textures tend to become anisotropic (Kranz, 1979). This in turn induces a strong elastic anisotropy, an effect which is not present in other materials such as metals where the main damage mechanisms (void nucleation and growth) are essentially isotropic. In previous models of damage, the elasticity tensor $C$ has been typically related in some way to the state of microcracking of the material, as described by some suitable set of softening parameters. By contrast, in the present model it is proposed that the values of the elastic compliances themselves be taken as a characterization of the state of damage of the material. The damage theory outlined below provides a set of equations of evolution for $C$. This formulation allows for general induced elastic anisotropy and limits the data base requirements to a minimum during computations.

It is also observed experimentally that the process of damage in concrete is not perfectly brittle. Thus, when the loads are removed the microcracks do not close completely and permanent or plastic strains remain. Materials exhibiting this type of behavior have been frequently termed plastic-fracturing. Finally, extended microcracks in concrete appear to soften the material not only in the direction normal to the cracks but in the in-plane directions as well. A familiar example is provided by the uniaxial compression test. In this case, the extended microcracks run on the average along the axis of loading. Experimental measurements show that the loss of stiffness of the specimen in the axial direction is substantial (Karsan and Jirsa, 1969).

In the special case of proportional monotonic loading the proposed model takes the following form:

(i) **Elastoplastic kinematics**:

$$\epsilon = \epsilon^e + \epsilon^p.$$  \hspace{1cm} (3.10)

(ii) **Elastic response**:

$$\epsilon^e = C : \sigma.$$  \hspace{1cm} (3.11)

(iii) **Elastic degradation**:

$$C = C^0 + C^d.$$  \hspace{1cm} (3.12)

(iv) **Plastic flow rule**:

$$\dot{\epsilon}^p = \mathcal{A} \dot{\mu} (\sigma^+ + c \sigma^-).$$  \hspace{1cm} (3.13)

(v) **Damage rule**:

$$C^d = (1 - \alpha) \hat{\mu} \left( \frac{\sigma^+ \otimes \sigma^+}{\sigma^+ : \sigma^+} + \frac{\sigma^- \otimes \sigma^-}{\sigma^- : \sigma^-} \right).$$  \hspace{1cm} (3.14)

(vi) **Damage and loading/unloading criteria**:

$$\dot{\mu} > 0 \quad \text{if} \quad \Phi(\sigma, \mu) = \sqrt{\sigma^+ : \sigma^+ + c \sigma^- : \sigma^-} - t(\mu) = 0,$$
$$\text{and} \quad (\sigma^+ + c \sigma^-) : D : \dot{\epsilon} > 0,$$
$$\dot{\mu} = 0 \quad \text{otherwise}.$$  \hspace{1cm} (3.15)

(vii) **Softening law**:

$$t(\mu) = f_n e^{\log(1 + E_0 \mu)}.$$  \hspace{1cm} (3.16)

In these expressions $\sigma$ is the stress tensor of mortar (denoted $\sigma_i$ in section 3.1) and $\sigma^+$ and $\sigma^-$ are the positive (tensile) and negative (compressive) parts of $\sigma$. These are defined as follows. Let $\sigma^{(a)}$ and $d^{(a)}$, $a = 1, 2, 3$ be the eigenvalues and eigenvectors of $\sigma$, respectively, so that

$$\sigma_{ij} = \sum_{a=1}^{3} \sigma^{(a)} d_i^{(a)} d_j^{(a)}.$$  \hspace{1cm} (3.17)

Then

$$\sigma_{ij}^+ = \sum_{a=1}^{3} \langle \sigma^{(a)} \rangle d_i^{(a)} d_j^{(a)}, \quad \sigma_{ij}^- = \sigma_{ij} - \sigma_{ij}^+.$$  \hspace{1cm} (3.18)
where $(\cdot)$ signifies the McAuley bracket, with its usual definition
\[
\langle x \rangle = \frac{1}{2}(x + |x|).
\] (3.19)

The tension-compression decomposition (3.18) of \( \sigma \) plays a similar role to that played by the hydrostatic-deviatoric decomposition in classical plasticity. In the present context, the underlying concept is that damage is driven both by tensile and compressive stresses, although the mechanisms whereby the latter come into play are less effective in causing damage than those associated with tension. In the extreme case in which damage is caused solely by tensile stresses the constitutive relations can be formulated in terms of \( \sigma^+ \) alone.

This situation parallels that encountered in classical isochoric plasticity in which the response functions are assumed to depend solely on the deviatoric part of stress. However, in the case of concrete the available experimental evidence shows that damage can occur under states of pure compression (Karsan and Jirsa, 1969). Thus, the compressive part of stress \( \sigma^- \) has to be included in the constitutive relations, though possibly carrying a lesser weight than \( \sigma^+ \).

Equation (3.10) expresses the commonly assumed additive decomposition of strain into elastic and plastic components. Much in the same spirit, it is assumed that the elastic compliances \( C_{\varepsilon} \) have an additive structure and can be expressed as the sum of the elasticity tensor \( C_{\varepsilon}^0 \) of the uncracked material and the added flexibility \( C_{\varepsilon}^d \) due to damage, (3.12). This assumed form of \( C \) is in line with self-consistent calculations of the overall elastic compliances of elastic media with distributed cracks (Budiansky and O'Connell, 1976; Horii and Nemat-Nasser, 1983; Kachanov, 1980).

Equations (3.13) and (3.14) are a set of rate-independent equations of evolution for plastic strains and damage. The rate at which the inelastic processes progress is determined by the multiplier \( \dot{\mu} \) and their direction by \( (\sigma^+ + c \sigma^-) \) for plastic flow and by \( (\sigma^+ \otimes \sigma^+/\sigma^+ : \sigma^+ + c \sigma^- \otimes \sigma^-/\sigma^- : \sigma^-) \) for damage, where \( c \) is a material parameter. Here, the symbol \( \otimes \) is used to denote the dyadic product, i.e., \( (a \otimes b)_{ijkl} = a_{ij} b_{kl} \). More general forms of flow and damage rules are given by Ortiz (1985). It is interesting to note that both the tensile and compressive parts of the stress tensor are assumed to play a role in furthering damage. The parameter \( c \) determines the extent of damage induced by compression relative to that induced by tension. Comparisons with experimental data indicate that \( c \) takes a value around 0.04, which renders tension the dominant driving force for damage. It is also noted that for arbitrary stress paths the damage rule (3.14) in general results in a strong elastic anisotropy. For proportional loading, the elastic compliances inherit the symmetries of the loads. For instance, in uniaxial compression the computed elasticity tensor is transversely isotropic.

The rationale behind the choice (3.13) of flow rule is the following. If the parameter \( \alpha \) is set to zero, the process of damage involves no plastic deformations, i.e., is perfectly brittle. The constitutive behavior so defined is such that no permanent strains are predicted to remain upon unloading of the material. Mortar, however, is known not to conform to this pattern (Adenaes et al., 1977). A sizable process zone, together with misfits on the surface of the cracks prevent them from closing completely and, thus, permanent strains develop. The total rate of inelastic deformation is then given by
\[
\dot{\varepsilon}^i = \dot{\mathbf{C}} : \sigma + \dot{\varepsilon}^p = \dot{\varepsilon}^d + \dot{\varepsilon}^p.
\] (3.20)

Here \( \dot{\varepsilon}^d \) is the rate of inelastic deformation due to damage and \( \dot{\varepsilon}^p \) is the plastic strain rate. Clearly, the orientation of the plastic strains is determined by that of the extended microcracks. Therefore, it seems reasonable to assume that both the inelastic strain rates \( \dot{\varepsilon}^d \) associated with microcrack extension and the plastic strain rates \( \dot{\varepsilon}^p \) are coaxial. A simple model of plastic fracturing is obtained by further assuming that the rate of plastic deformation is a constant fraction of the total inelastic strain rate, i.e., \( \dot{\varepsilon}^p = \alpha \dot{\varepsilon}^i \), where \( \alpha \) is a material parameter. Equations (3.13) and (3.14) readily follow from these assumptions. With this simple model, damage ranging from purely brittle (\( \alpha = 0 \)) to perfectly ductile (\( \alpha = 1 \)) can be accounted for.

In (3.13) and (3.14) the irreversible character of damage and plastic flow necessitates that
\[
\dot{\mu} > 0,
\] (3.21)
i.e., that damage and plastic deformations be ever increasing functions of time. The condition \( \dot{\mu} > 0 \) signals that the inelastic mechanisms are active. On the other hand, \( \dot{\mu} = 0 \) implies elastic behavior. A criterion is needed to ascertain under what conditions further inelasticity occurs, i.e., \( \dot{\mu} > 0 \), and when the material remains elastic, i.e., \( \dot{\mu} = 0 \). In classical plasticity, yield and loading criteria serve precisely this purpose. In the present context, a similar role is played by the damage and loading criteria (3.15). These conditions are amenable to the following geometric interpretation, Fig. 1. The locus of points in stress space such that \( \Phi = 0 \) may be viewed as a damage surface enclosing an elastic domain within which the response of the material is elastic. For inelasticity to take place, two conditions must be simultaneously satisfied: (i) The stress point must lie on the damage surface, and (ii) the trial stress increment \( D : \dot{\epsilon} \) must point outside the elastic domain.

The variable \( t(\mu) \) in (3.15) and (3.16) is the critical stress for the onset of damage in pure tension. For the model at hand, it is possible to determine the dependence of the critical stress \( t \) on the cumulative damage parameter \( \mu \) directly from the uniaxial tension test. Thus, particularizing the stress–strain relations to uniaxial tension conditions one obtains

\[
\sigma = \frac{\epsilon}{1/E_0 + \mu} \tag{3.22}
\]

where \( E_0 \) is the initial Young's modulus. Let \( \sigma(\epsilon) \) denote the experimentally determined uniaxial tension stress–strain curve. Replacing \( \sigma \) in (3.22) by \( \sigma(\epsilon) \), it is possible to solve for \( \epsilon \) as a function of \( \mu \), i.e., \( \epsilon = \epsilon(\mu) \), which together with the stress–strain relation \( \sigma(\epsilon) \) finally yields

\[
t(\mu) = \sigma(\epsilon(\mu)). \tag{3.23}
\]

It is therefore concluded that the softening law is completely determined by the uniaxial tensile test. This situation parallels that encountered in isotropic plasticity where the hardening law follows also from the uniaxial test. A convenient expression for the stress–strain curve of mortar in uniaxial tension is given by the equation

\[
\sigma = f_t \frac{\epsilon}{\epsilon_t} \exp \left( 1 - \frac{\epsilon}{\epsilon_t} \right) \tag{3.24}
\]

proposed by Smith and Young (1955) in the context of uniaxial compression. In the present context, the parameters \( f_t \) and \( \epsilon_t \) signify the tensile strength and peak strain in tension, respectively. The resulting softening law is the one given in (3.16).

### 3.3. Aggregate as a granular medium

In this section a simple model for aggregate is discussed which combined with the damage model for mortar presented in section 3.2 and the equations of mixture theory summarized in section 3.1 completes the formulation of the proposed model for concrete. As in section 3.2, for simplicity of notation the subindex 2 pertaining to aggregate as the second phase of concrete is dropped throughout this section.

The behavior of granular media exhibits a number of complexities which have been the subject of extensive research (see e.g., Nemat-Nasser, 1983, for a review). Noteworthy among these are dilatancy and non-coaxiality effects. However, in concrete the deformation of aggregate is severely restrained by mortar. Thus, in this context it is doubtful that many of the subtleties in the behavior of aggregate as a granular material play a significant role. It seems therefore appropriate to use a simple model to describe the behavior of aggregate as a constituent of concrete. A criterion...
which has been frequently used to characterize the failure of cohesionless soils is the Drucker–Prager failure criterion (Drucker and Prager, 1952) which reads

$$\Phi(\sigma) = q -Mp$$  \hspace{1cm} (3.25)

where $p = \frac{1}{3} \sigma_{kk}$ is the hydrostatic pressure, $q = (\frac{1}{2} s_{ij} s_{ij})^{1/2}$, with $s_{ij} = a_{ij} - \rho \delta_{ij}$, is the effective stress deviator and the internal friction coefficient $M$ is related to the internal friction angle. Furthermore, it is well-established that in the presence of internal friction, plastic flow is nonassociated (Drucker, 1950; 1959; Mandel, 1966). This is yet another distinct characteristic of granular media which manifests itself as the fact that the dilatancy coefficient $N$, defined as the ratio between volumetric and deviator plastic strain rates, does not in general coincide with the internal friction coefficient $M$ (see e.g., Atkinson and Bransby, 1978).

This motivates considering a flow potential of the type

$$\Psi(\sigma) = q - Np$$ \hspace{1cm} (3.26)

where in general $N \neq M$. The plastic strain rates are then given by

$$\dot{\epsilon}^p = \mu \frac{\partial \Psi(\sigma)}{\partial \sigma}$$ \hspace{1cm} (3.27)

and, since $\Psi \neq \Phi$, the plastic flow is nonassociated. The flexibility of having independent material parameters governing failure and dilatancy has been found of primary importance in the course of checking the predictions of the model against experimental data.

4. Localized failure modes of concrete

In this section, the localization techniques discussed in Section 2 are combined with the constitutive model for concrete summarized in Section 3, and the localized failure modes for various biaxial stress paths are computed. The material parameters utilized throughout the calculations are:

(i) Mortar: Volumetric fraction $\alpha_1 = 0.5$. Initial Young’s modulus $= 5000$ ksi. Initial Poisson’s ratio $= 0.2$. Tensile strength $f_t = 0.8$ ksi. Peak strain in tension $\epsilon_t = 0.4 \times 10^{-3}$. Cross-effect coefficient $c = 0.04$. Plastic-fracturing coefficient $\alpha = 0.4$.

(ii) Aggregate. Volumetric fraction $\alpha_2 = 0.5$. Initial Young’s modulus $= 5000$ ksi. Initial Poisson’s ratio $= 0.2$. Internal friction angle $= 30^\circ$ ($M = 1.2$). Dilatancy angle $= 10^\circ$ ($N = 0.37$).

The uniaxial tension and compression stress–strain curves corresponding to this choice of parameters are shown in Fig. 2.

The results presented below were obtained assuming linear stress paths and straining the material monotonically up to failure. Attention was confined to biaxial states of stress. The stress–strain equations for concrete (3.8) were integrated using forward-Euler’s explicit algorithm. At each step during the integration process the
current tangent stiffness $D^{(t)}$ was computed and the localization condition (2.13) checked. The analysis predicts different failure modes depending on whether the loading path is one of biaxial compression, uniaxial compression, compression-tension, tension-compression, uniaxial tension or biaxial tension. These cases are examined next and the results of the analysis contrasted with experimental data.

4.1. Biaxial compression

Analyses involving biaxial compression resulted in localized failure. Localization occurred just prior to the peak in the stress-strain curve for stress ratios ranging from $\sigma_1/\sigma_2 = -1/-1$ approximately $\sigma_1/\sigma_2 = -4/-1$. For stress paths closer to uniaxial compression, localization occurred after the peak. The strain at localization increased steadily relative to the peak strain as the stress path approached uniaxial compression. In this limiting case, no localization was detected (see section 4.2).

The localized failure mode for biaxial compression predicted by the method is shown in Fig. 3. The plane of failure contains the smaller compression direction and makes an angle $\phi$ with the largest compression direction. This result is in keeping with experimental observations by van Mier (1984), Rosenthal and Glucklich (1970), Kupfer, Hilsdorf and Rusch (1969) and Robinson (1967). The values of $\phi$ computed for various biaxial stress paths are given in Table 1. It is observed that $\phi$ takes its smallest value $\phi = 25^\circ$ for symmetric biaxial compression $\sigma_1/\sigma_2 = -1/-1$ and increases monotonically to $\phi = 35^\circ$ for $\sigma_1/\sigma_2 = -16/-1$. Van Mier measured values of $\phi = 21^\circ$ for $\sigma_1/\sigma_2 = -1/-0.33/-0.05$ and $\phi = 24.5^\circ$ for $\sigma_1/\sigma_2 = -1/-0.10/-0.05$. These values of $\phi$ and the computed ones exhibit the same increasing trend as the stress path approaches uniaxial compression. Kupfer, Hilsdorf and Rusch reported values in the range 18-27° for biaxial compression.

The computed values of the angle $\psi$ defined in (2.5) are also given in Table 1. As may be seen, $\psi$ takes its lowest value $\psi = 56.45^\circ$ for $\sigma_1/\sigma_2 = -1/-1$ and steadily increases towards $\psi = 90^\circ$ as the loading path approaches uniaxial compression. Thus, failure under symmetric biaxial loading is predicted to involve more separation or cleavage than failure under predominantly uniaxial compression conditions. For stress paths near the uniaxial compression axis a failure mode akin to a shear band is obtained. In all cases localized failure is seen to involve both separation and plastic flow in varying degrees.

4.2. Uniaxial compression

Under uniaxial compression loading no localization was detected. Thus, the method predicts that failure under uniaxial compression takes place in a distributed form. It should be emphasized, however, that the uniaxial compression path $\sigma_1/\sigma_2 = -1/0$ appears to be an isolated direction in this respect and that a slight variation from un-
iaxial conditions will result in localization. Thus, the paths $\sigma_1/\sigma_2 = -16/-1$ and $\sigma_1/\sigma_2 = -16/1$ exhibit localized failure (see sections 4.1 and 4.3). For these nearly uniaxial paths, failure is predominantly of shear type. Exact uniaxiality is difficult to obtain in the laboratory and thus many investigators report localized failure modes of the type predicted by the method for nearly uniaxial paths. For example, Kupfer, Hilsdorf and Rusch observed localized failure in uniaxial compression at an angle $\phi = 30^\circ$. However, more recent work has been able to detect distributed failure modes. For instance, van Mier carefully documents a case in which the fracture pattern exhibits a profusion of crisscrossing cracks with no apparent localization of damage (van Mier, 1984, p. 86). This evidence would appear to lend support to the conclusion that distributed failure takes place under mathematically exact uniaxial compression conditions.

4.3. Compression-tension

Under compression-tension conditions (i.e. for loading paths in the range $\sigma_1/\sigma_2 = -1/0$ through $\sigma_1/\sigma_2 = -1/1$), the analysis predicts the development of localized failure modes. The geometry of these modes is depicted in Fig. 4 and the values of the parameters for various loading conditions are given in Table 2. It is observed that there is a continuous transition from near-uniaxial compression to tension-compression type failure. Thus, for near-uniaxial compression paths (e.g., $\sigma_1/\sigma_2 = -16/1$) the failure planes arise at $\phi = 30^\circ$ and the localized modes have a shear band character ($\psi = 72.15^\circ$). As the tensile component of the loading is increased, the plane of failure tends to orient itself normal to the tensile principal direction. Concurrently, the angle $\phi$ steadily decreases in value, which is indicative of a trend towards splitting failure modes. By the time the diagonal of the compression-tension quadrant is reached, failure is purely tensile (see section 4.4). The existence of a smooth transition from compressive to tensile failure as the loading path ranges from
uniaxial compression to compression-tension has been determined experimentally (Isenberg, 1968; Kupfer et al., 1969). However, the limiting stress ratio for compressive-like failure has been measured at $\sigma_1/\sigma_2 = -15/1$ by Kupfer Hildorf and Rusch (1969) and at $\sigma_1/\sigma_2 = -17/1$ by Isenberg (1968), which is in contrast with the analytically determined ratio $\sigma_1/\sigma_2 = -1/1$.

4.4. Tension-compression and uniaxial tension

The analysis also detected localization under tension-compression conditions (i.e., for loading paths in the range $\sigma_1/\sigma_2 = -1/1$ through $\sigma_1/\sigma_2 = 0/1$). In this case, failure was purely tensile in character, i.e., separation band ($\psi = 0$) developed normal to the tensile principal direction, Fig. 5. This result is in agreement with numerous experimental observations (Carino and Slate, 1976; Kupfer et al., 1969; McHenry and Karni, 1958; Rosenthal and Glucklich, 1970).

4.5. Strains at localization

A question of some theoretical and practical interest concerns the strains at which localization takes place. These strains have proven to be extremely sensitive to the detailed structure of the constitutive relations.

Table 3 shows the computed strains $\varepsilon_{loc}$ at localization and compares them to the strains $\varepsilon_{max}$ at which the stress-strain curve peaks. The scalar strains $\varepsilon_{loc}$ and $\varepsilon_{max}$ displayed in Table 3 are to be understood as follows. If the stress path under consideration is of the type $(\sigma_1, \sigma_2) = \sigma(a_1, a_2)$, with $a_1^2 + a_2^2 = 1$, then the scalar strain for that path is defined as the strain work conjugate to the loading parameter $\sigma$, i.e., $\varepsilon = a_1 \varepsilon_1 + a_2 \varepsilon_2$.

It is seen from Table 3 that localization can occur both before and after the peak stress. Material instabilities in the ascending branch of the stress-strain curve are made possible by the assumed nonassociated character of plastic flow in the aggregate. It is also observed that localization is delayed as confinement is increased and as uniaxial compression conditions are approached. In the limit of perfect uniaxial compression the strain required for localization tends to infinity and failure takes place in a distributed fashion. By contrast, when the tensile principal strain becomes dominant localization is seen to coincide with the peak of the stress-strain curve.

5. Conclusions

A theoretical framework for the analysis of localized failure in concrete has been presented. The theory is predicated upon the assumption that discrete failure planes arise as a result of a process of localization of damage. The onset of localized modes can be viewed as a bifurcation phenomenon whereby local neighborhoods of the material depart from near-uniform straining in favor of highly localized deformation patterns. A simple bifurcation analysis suffices to detect when localization initiates and to determine the geometry of the localized deformation modes.

Localization techniques provide a simple yet effective means of extending the range of applicability of traditional distributed damage models to situations involving localized failure. The numerical results presented in this paper exhibit a good overall agreement with experimental observations. It should be emphasized that localization analyses have proved highly sensitive to variations in the material parameters and the details of the material model. Thus, considering that no effort has been made here to optimize the material parameters used in the analysis and that compari-
sons have been drawn with a variety of test data obtained from different concretes and experimental techniques. The qualitative agreement obtained is rather encouraging.

The fact that localization analyses are highly sensitive to the details of the constitutive equations renders them a valuable tool in assessing the accuracy of material models. In the context of the constitutive framework utilized in this paper, it was found that the use of mixture theory as a means of determining the effective stresses which drive damage in mortar was instrumental in being able to capture the wide variety of failure modes exhibited by concrete.

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