A CONSTITUTIVE THEORY FOR THE INELASTIC BEHAVIOR OF CONCRETE

Michael ORTIZ

Division of Engineering, Brown University, Providence, RI 02912, U.S.A.

Received 3 December 1984; revised version received 5 February 1985

A general theory for the inelasticity of concrete is proposed, the main constituents being a new, rate independent model of distributed damage for mortar and the application of mixture theories to account for the composite nature of concrete. The proposed theory of damage is capable of accommodating fully anisotropic elastic degradation, both in tension and in compression, in a manner which is ideally suited for computation. Mixture theories, on the other hand, are found to provide a simple yet effective tool for characterizing the values of the phase stresses that act on mortar and aggregate and which drive damage and plastic flow. This uneven distribution of stresses between mortar and aggregate is seen to lie at the foundation of effects such as the characteristic splitting failure modes in uniaxial compression and the unloading hysteretic loops that arise during cyclic loading. Further to furnishing useful insights into the physical mechanisms underlying the inelastic behavior of concrete, the proposed model provides a simple means of quantifying such behavior in a way which can be readily implemented in any standard finite element code. Possible generalizations of the theory are suggested. In particular, it is noted how rate and rheological effects can be incorporated into the proposed framework by extending it into the viscoplastic range and through the use of Eyring's theory of thermal activation.

1. Introduction

Nonlinear analysis of reinforced concrete structures has become increasingly important in recent years. It is only by carrying out an incremental inelastic analysis of the structures up to collapse that it is possible to assess with some degree of certainty all safety aspects and deformational characteristics. This type of analysis is particularly desirable for certain structures such as concrete reactor vessels, nuclear containment and offshore platforms for which experimental studies are often prohibitively expensive.

With the present state of development of computer codes based on the finite element method, inadequate modeling of concrete is probably the major factor limiting our computational capabilities. Despite the widespread use of concrete as a structural material, our knowledge of its physical properties and behavior under arbitrary loading is rather deficient, let alone our ability to quantify such behavior. This most likely owes to the stupendous variety and complexity of the observed material phenomenology. Effects such as anisotropic elastic degradation, nonassociated plastic flow, unstable post-failure behavior and hysteretic unloading loops, to mention only a salient few, turn the task of developing constitutive models for concrete for all stages of loading into one of the most difficult challenges to be faced today by researchers in the field.

Unlike metals, concrete is a strongly heterogeneous material which exhibits several mutually interacting inelastic mechanisms such as microcrack growth and plastic flow. Furthermore, the microstructure of concrete appears far more random and chaotic than the harmonious crystal lattices of metals. This explains why there has not existed to present date a physical theory of the inelasticity of concrete that could play a role similar to that of dislocation mechanics in polycrystalline research. As a consequence, constitutive models for concrete have been predominantly phenomenological in nature, rather than being based directly on first principles of mechanics. The feeling is gradually growing strong, however, that this situation is not ideal. As in the case of metals, concrete research would undoubtedly benefit
greatly from the unifying thrust of a solid and sound physical model.

In summary, for a constitutive theory of concrete to fully meet today’s needs and standards it should comply with three basic requirements:

- It should be based on first principles of mechanics and provide a coherent and comprehensive physical explanation for the observed experimental phenomenon. This includes such effects as the development of distributed cracks and ensuing degradation of elastic properties, the nonassociated, unstable plastic response of the material and the hysteretic unloading loops.
- It should be capable of accurately reproducing a broad sample of experimental records, both uniaxial and multiaxial, under monotonic and cyclic loading, including proportional and non-proportional stress paths in the tensile and compressive ranges.
- It should be suitable for use in computation. In particular, the model should lend itself to a straightforward and efficient implementation in finite element codes.

In this paper, a new constitutive theory for concrete is outlined which shows considerable potential for satisfying the aforementioned requirements. The main constituents of the theory are a new, rate independent model of distributed damage for mortar and the application of mixture theories to account for the composite nature of concrete. The proposed theory of damage is capable of accommodating fully anisotropic elastic degradation, both in tension and in compression, in a manner which is particularly convenient for computation. Mixture theories, on the other hand, are found to provide a simple yet effective means of characterizing the values of the phase stresses that develop in mortar and aggregate and which drive the inelastic processes. This uneven distribution of stresses between mortar and aggregate is seen to lie at the foundation of effects such as the characteristic splitting failure modes in uniaxial compression and the unloading hysteretic loops that arise during cyclic loading.

Following a brief review in Section 2 of the available body of knowledge concerning the inelastic behavior of concrete and the underlying physical mechanisms, the main features of the theory are presented in Section 3. Some tests of the model are shown in Section 4 which in the light of the available experimental data illustrate how the theory adequately predicts, among other effects:

- The development of microcrack patterns and failure modes, both in tension and in compression and under multiaxial states of stress.
- The degradation of the elastic response of the material and induced elastic anisotropy.
- The marked difference in tensile and compressive strengths.
- The strong positive dilatancy and unstable post-failure behavior.
- The stabilizing effect of lateral confinement.
- The unloading hysteretic loops.

In conclusion, the potential benefit to be derived from the use of mixture theories is that simple models for both mortar and aggregate are capable of capturing most of the complex phenomenology of concrete. Furthermore, mixture theories provide useful insights into the physical nature of the inelastic mechanisms underlying the behavior of concrete, while furnishing a simple means of quantifying such behavior in manner suitable for computation.

2. Material behavior of concrete and underlying inelastic mechanisms

There exists today a sizable body of experimental evidence which rather conclusively points to two physical mechanisms and their interaction as the underlying causes for much of the phenomenology of concrete. These mechanisms are:

- Microcrack growth, and
- Slip-type plastic flow.

The extension of microcracks, for instance, is known to play a decisive role in the inelasticity of concrete, as it results in the degradation of the elastic compliances (Hsu et al., 1963; Gardner, 1969; Karsan and Jirsa, 1969; Mills and Zimmerman, 1970; 1971; Linse, 1973; Palaniswamy and Shah, 1974; Wastiels, 1979) and interacts with the plasticity of the material (Hueckel, 1975; 1976; Hueckel and Maier, 1977; Dafalias, 1977a; 1977b; 1978), an effect which is known as elastoplastic coupling.

It is important to note, however, that both the
cracking and plastic flow of concrete exhibit a variety of atypical features that are not contained within the classical theories of fracture mechanics and plasticity. It is a well-known fact, for instance, that when concrete is subjected to uniaxial compression it develops cracks that are parallel to the axis of loading (see, e.g., Wastiels, 1979 and references contained therein). In some cases, these cracks become so large as to be the direct cause of failure of the specimen. This situation persists even if the specimen is laterally confined by means of a moderate compressive pressure. It is thus concluded that cracks in concrete can open against compressive stresses, which is in apparent contradiction to the second law of thermodynamics that requires that cracks open only under tension (see, e.g., Sneddon, 1969).

Early explanations of this effect idealized the situation as that of an infinite medium (mortar) with a stiff spherical inclusion (Taylor and Broms, 1964; Shah and Winter, 1966). When an overall uniaxial compressive loading is imposed upon this system, a linear elastic analysis reveals that tensile stresses tend to appear around the particle at right angles with the direction of loading, which eventually may cause longitudinal cracking. This model highly idealizes the problem, as it entirely neglects the effect of neighboring aggregate particles and does not appear to have been pursued further. On the other hand, systems of periodically or randomly distributed inclusions have been considered (Nemat-Nasser and Taya, 1981; Nemat-Nasser, Iwakuma and Hejazi, 1982; Suquet, 1982; Horii and Nemat-Nasser, 1983). In spite of the strong theoretical appeal of this approach, the complexity of the analyses is such, particularly in three dimensions and in the presence of extended microcracking, that it is not yet clear how these theories may serve as a basis for devising material laws for concrete.

A simple yet general theoretical framework that offers a plausible explanation for the cracking modes that are peculiar to concrete and which appears to be in good keeping with the experimental observations has been proposed by Ortiz and Popov (1982a). The theory acknowledges from the outset that concrete is a composite material comprising two main phases: mortar and aggregate.

This is in sharp contrast to most models proposed in the past, which treat concrete as a simple or homogeneous material. It is demonstrated by Ortiz and Popov how the theory of interacting continua can be applied to concrete to conclude, for instance, that the applied stresses distribute unequally between mortar and aggregate, and to quantify in a simple manner the induced phase stresses as a function of the applied ones. In particular, a state of compressive uniaxial loading is seen to result in significantly large transverse splitting stresses in mortar. This in turn provides an explanation for the longitudinal microcracks which are observed to develop during the compressive uniaxial test. Physically, the situation is illustrated in Fig. 1. The tendency of the aggregate particles to move apart sideways is the cause for the tensile lateral stresses appearing in mortar.

In sum, the composite nature of concrete seems to have a decisive influence on the development and propagation of microcracks. On the basis of these recent theoretical studies (Ortiz and Popov, 1982a; 1982b), it appears that the theory of interacting continua provides a simple yet effective means of characterizing this aspect of the material. At any rate, it seems reasonable to make the detailed study of the consequences of the composite nature of concrete the first step towards

---

Fig. 1. Mixture theoretical interpretation of splitting failure mode. When concrete is acted upon by uniaxial compressive stresses, aggregate tends to be squeezed out sideways. This introduces high splitting stresses in mortar which in turn induce extended cracking along the axis of loading.
devising adequate, physically based constitutive models for concrete.

Another salient aspect of the behavior of concrete is the process of damage undergone by its elastic properties as a consequence of microcrack growth (Gardner, 1969; Karsan and Jirsa, 1969; Linse, 1973; Mills and Zimmerman, 1970; 1971; Palaniswamy and Shah, 1974; Wastiels, 1979). It has been experimentally shown through crack surveys that crack textures quickly become highly anisotropic (Kranz, 1979). This endows the elasticity of concrete with a strong induced anisotropy. Despite this well-known fact, most damage models proposed in the past are isotropic (Kupfer and Gerstle, 1973; Budiansky and O'Connell, 1976; Cedolin et al., 1977; Resende and Martin, 1984; Bazant and Kim, 1979) or orthotropic (Liu, Nilson and Slate, 1972a; 1972b; Romstad, Taylor and Herrman, 1974; Darwin and Pecknold, 1977; Elwi and Murray, 1979; Bashur and Darwin, 1978). This clearly restricts the validity of such models to certain loading conditions.

A few theories are available that do account for induced anisotropy (Dougill, 1976; Dougill et al., 1977; Dougill and Rida, 1980; Kachanov, 1980; Costin and Holcomb, 1981, 1983; Ortiz and Popov, 1982a; Costin, 1983a; 1983b; Horii and Nemat-Nasser, 1983). Some of these theories have characterized microcrack textures by controlling microcrack growth along selected directions. To obtain a faithful representation of the process of damage, however, numerous such control directions need to be considered. This in turn necessitates extensive data bases which render the models unfit for structural computations. Other theories are based on self-consistent calculations which add considerably to the cost of numerical analyses.

This paper is concerned, to a large extent, with a new theory of anisotropic damage which overcomes these difficulties. The theory would appear to constitute the natural extension of classical plasticity to materials which exhibit elastic degradation, and is ideally suited for computation. Although some of the features of the model were present in previous work by the author (Ortiz and Popov, 1982b), the formalism herein proposed is novel in that it takes the current values of the elastic compliances of the material as the measures of damage. Thus, the elasticity tensor of the material is regarded as an internal variable representing the state of microcracking. This has the effect of limiting the data base requirements of the model to a minimum, since it is precisely the current value of the elastic compliances which is directly relevant to computation. Furthermore, no particular elastic symmetries need to be assumed and fully anisotropic behavior can be accounted for.

The proposed theory of damage hinges around the notion of a rate-independent damage rule, which in effect parallels the role played by flow rules in classical plasticity. Damage rules define the direction in which the instantaneous elastic degradation takes place, very much like flow rules determine the current direction of plastic flow. Once a particular form of damage rule is postulated, suitable criteria for the onset of damage follow directly from the second law of thermodynamics (Lubliner, 1975; 1978: 1980; Ortiz and Popov, 1982b). This defines a damage threshold below which further damage occurs, much in the same fashion as yield criteria define elastic domains within which plastic flow cannot take place.

The proposed damage model is, to a certain extent, phenomenological. Just as hardening rules in classical plasticity need to be identified from experiment, a softening rule is imbedded within the formulation which must be identified on the basis of the tensile uniaxial stress–strain curve of the material. In particular, such stress–strain curve can be reproduced exactly by the model.

In the context of concrete as a mixture, damage may be conveniently viewed as taking place in mortar and to be driven by the phase stresses acting on it. In general, as mentioned above, such phase stresses depart significantly from the applied ones. In particular, tensile splitting stresses may develop in mortar when the applied loading is entirely compressive, such as in the uniaxial compression test. Thus, damage may possibly progress even when the loading is not tensile, as observed in practice. This is in sharp contrast with most models proposed heretofore which can only quantify damage under purely tensile conditions.

The softening effect of damage in the overall material response has several potentially far-re-
aching consequences concerning the nature of the associated boundary value problem. Foremost among these is the loss of ellipticity with the ensuing possibility of localization of the inelastic deformation. While the theory of localization is presently well developed (Rice, 1976; Needleman and Rice, 1978), it has mainly been applied to the study of shear band formation in metals and soils. The case of concrete presents an interesting variant of the process of localization, namely, the localization of diffuse microcracking into discrete cracks. Such extended cracks are observed to develop in reinforced flexural members and an efficient quantification of parameters relevant to design such as crack spacing and opening width would be of considerable value.

The plastic response of concrete, or the development of plastic or irrecoverable strains, also exhibits a number of features which are foreign to the classical theory of plasticity. For instance, lack of conformity with the normality rule has been shown experimentally by Adenaes et al. (1977). On the other hand, the characteristic descending branch of the uniaxial stress–strain diagram of concrete has been commonly viewed as a violation of Drucker's stability postulate. Considerable effort has been devoted in the past to extending the classical theory of plasticity to a framework suitable for the study of such materials as concrete, rocks and granular media. Weak stability criteria have been proposed that relax the requirements of Drucker's postulates and allow for unstable behavior (Il'iuishin, 1961; Nguyen and Bui, 1974; Bazant, 1980). However, this work is mostly speculative and does not address the issue of why concrete appears to violate the classical stability postulates. A further complication arises from the unloading hysteretic loops that develop when concrete is subjected to cyclic loading (Karsan and Jirsa, 1969; Sinha, Gerstle and Tulin, 1964; Spooner and Dougill, 1975). These unloading loops have heretofore defied explanation. Furthermore, their numerical modeling in the context of plasticity has frequently involved questionable artifacts such as internal variables which experience sudden jumps in time (Bazant and Kim, 1979).

In the model of concrete herein proposed, plastic or irrecoverable strains arise from two independent sources: plastic microcracking and plastic flow of aggregate. The fact that the microcracking of concrete in not perfectly brittle has been noted before (Bazant 1983). To some extent, the opening of microcracks is irreversible and results in the development of permanent or plastic strains. On the other hand, aggregate is a granular material and as such is capable of undergoing extensive plastic flow. The overall plastic deformation of concrete is thus characterized as the aggregate effect of these two mechanisms, in a mixture theoretical sense to be made precise below.

The proposed model warrants the following general conclusions concerning the plastic response of concrete:

- The lack of normality of the plastic strain rates is inherited from the nonassociativity of plastic flow in aggregate. Nonassociativity or lack of normality is in fact a distinctive feature of the plasticity of granular media (Atkinson, Bransby, 1978; Nemat-Nasser and Shokooh, 1980). More generally, the normality rule is known to loose its validity in the presence of internal friction (Drucker, 1950; 1959; Mandel, 1966).

- The unstable or softening stage is a manifestation of the process of damage undergone by the material rather than being due to instabilities in the plastic response. In fact, the models used below for plastic microcracking and the plastic flow of aggregate are stable in the classical sense and yet a strongly softening stage is predicted.

- The unloading hysteretic loops appear as a natural and direct consequence of the composite nature of concrete. It is shown below that the process of plastic deformation results in high, mutually equilibrating residual stresses in mortar and aggregate. In the latter, these residual stresses suffice to cause yielding upon unloading of the externally applied stresses. This yielding manifests itself in the development of sizeable unloading loops in the overall response of concrete.

This last result emphasizes once more the preeminent role played by the composite nature of concrete in shaping its material behavior. In fact, the theory outlined next vividly illustrates that by acknowledging from the outset the basic fact that concrete is a mixture rather than a simple material, simple models for both mortar and aggregate
suffice to predict all of the complex phenomenology observed experimentally.

3. Outline of the proposed model

In this section, a constitutive model for the inelastic behavior of concrete is outlined. It would appear that this model has considerable potential for meeting the requirements mentioned in the introduction, i.e., being physically plausible, accurately fitting experimental data and being suitable for computation.

The proposed model rests upon the assumption that the inelastic behavior of concrete is the result of three concurring factors:

- The material behavior of mortar.
- The material behavior of aggregate and
- The interaction between mortar and aggregate.

Thus, concrete is viewed from the outset as a composite material or mixture comprising two phases: mortar and aggregate. In this context, attention needs to be given, first, to the problem of modeling mortar and aggregate as separate materials. Once this is achieved, the overall behavior of concrete can be predicted with the aid of the theory of interacting continua or theory of mixtures.

This is, in fact, one of the major points of departure of the present theory with regard to most presently advocated models which regard concrete as a simple or uniform material. As becomes apparent below, the pay off to be derived from this new strategy is significant. Indeed, by using simple models for both mortar and aggregate and combining them by means of a simple mixture theory, all of the complex phenomenology of concrete arises naturally.

In the remaining of this section two material models for aggregate and mortar are presented. For the former, a new rate independent damage model is proposed which is capable of characterizing anisotropic elastic degradation effects in a manner that would appear ideally suited for computation. For the latter, the use of the simple Mohr–Coulomb plasticity model for granular materials is advocated. Finally, a mixture theory essentially due to Green and Naghdi (1965) is applied to the case at hand to derive a set of constitutive relations for concrete.

3.1. A rate independent damage model for mortar

We now turn our attention to the problem of modeling the process of damage undergone by mortar. We start by recalling that the most general form of the Gibbs energy potential for a brittle material with a continuous distribution of microcracks can be shown to be (Ortiz and Popov, 1982a)

\[ G = \frac{1}{2} \sigma : C : \sigma - A^c \]  (3.1)

where \( \sigma \) signifies the stress tensor, \( C \) is the elasticity tensor of the material and \( A^c \) is the free energy required to form the microcracks, e.g., the surface energy of the cracks. Here, the symbol (: ) denotes the dyadic product, e.g., \( (C : \sigma)_{ij} = C_{ijkl} \sigma_{kl} \). Throughout this section it should be born in mind that the stresses acting on mortar, herein denoted by \( \sigma \), may be significantly different from the externally applied ones when mortar is a constituent of concrete. The relation between the two states of stress is then given by an influence or concentration tensor. In the present theory, such influence tensors are determined with the aid of mixture models. A detailed discussion of this aspect of the theory is included in section 3.3. Also, it should be noted that, unlike in the case of elastic or elastoplastic materials, the elastic moduli are now strongly structure sensitive, i.e., dependent on the state of microcracking of the material. As a result, \( C \) is no longer a material constant but will in general experience an evolution during a process of loading.

Following Coleman and Gurtin (1967), the stress–strain relations that correspond to the form (3.1) of energy potential are simply given by

\[ \epsilon = \frac{\partial G}{\partial \sigma} = C : \sigma. \]  (3.2)

The rate form of these relations is in turn given by

\[ \dot{\epsilon} = C : \dot{\sigma} + \dot{C} : \sigma = \dot{\epsilon}^e + \dot{\epsilon} \]  (3.3)

where \( \dot{\epsilon}^e \) is the elastic rate of deformation, i.e., that deformation rate that would be obtained by preventing the microcracks from extending further
and $\dot{\epsilon}^i$ is the inelastic rate of deformation due to the degradation of the elastic properties of the material.

In previous models of damage, the elasticity tensor $C$ has been typically related in some way to the state of microcracking of the material (Kachanov, 1980; Costin and Holcomb, 1981; 1983; Ortiz and Popov, 1982b; Costin, 1983a; 1983b; Horii and Nemat-Nasser, 1983). In the present model, by contrast, it is proposed that the values of the elastic compliances themselves be taken as a characterization of the state of damage of the material. This formulation allows for induced elastic anisotropy and limits the data base requirements to a minimum. In particular, it is assumed that the elastic compliances have an additive structure

$$C = C^0 + C^c,$$  \hspace{1cm} (3.4)

in terms of the elasticity tensor $C^0$ of the uncracked material and the added flexibility $C^c$ due to the active microcracks, i.e., to the microcracks that are acted upon by tensile stresses and are in the process of opening. The assumed form of $C$ is in line with self-consistent calculations of the overall elastic compliances of elastic media with distributed cracks (Budianski and O'Connel, 1976; Kachanov, 1980; Horii and Nemat-Nasser, 1983).

Clearly, the set of active microcracks will be a function of the state of stress, and hence so will be $C^c$. A general methodology for rendering this effect in mathematical terms is given next.

### 3.1.1. Closing of microcracks

Substituting (3.4) into (3.2) one obtains

$$\epsilon = (C^0 + C^c) : \sigma = \epsilon^0 + \epsilon^c$$  \hspace{1cm} (3.5)

where $\epsilon^0$ is the deformation that would occur in the absence of microcracking and $\epsilon^c$ is the deformation due to the microcracks. It may happen that some of the microcracks that are present in the material are at a given time acted upon by compressive stresses and remain closed, thereby not contributing to $\epsilon^c$. The closing of cracks may result in a sudden stiffening of the material, an effect that appears to lie behind the characteristic spindle shaped hysteretic loops that are observed experimentally in flexural members subjected to cyclic loading (Bertero and Popov, 1975).

A mathematically convenient way of expressing this situation is to require that $\epsilon^c$ have always positive eigenvalues, i.e., that it be positive definite. This insures that in any given direction, the deformation introduced by $\epsilon^c$ will be always positive, i.e., it will correspond to stretching due to opening of microcracks. The set of all positive definite tensors is commonly termed the positive cone and denoted by $C^+$. Thus, the opening condition may be viewed as a unilateral constraint that requires that $\epsilon^c$ be contained within the positive cone. Fig. 2. Symbolically,

$$\epsilon^c \geq 0.$$  \hspace{1cm} (3.6)

It follows from standard results in the theory of nonlinear optimization (Duvaut and Lions, 1976) that, for a given state of stress $\sigma$, the corresponding state of strain which is consistent with the closing condition (3.6) satisfies the constrained minimization problem

minimize $\frac{1}{2} \epsilon : (C^0 + \widetilde{C^c})^{-1} : \epsilon - \sigma : \epsilon,$  \hspace{1cm} (3.7)

subject to $\epsilon^c = \epsilon - C^0 : \sigma \geq 0,$

where $\widetilde{C^c}$ denotes the added flexibility that would be obtained if all of the microcracks were active. In order to characterize the solution to problem (3.7), it proves convenient to introduce the orthogonal projection $P^+$ of strain space onto the positive cone $C^+$. This operator assigns to every state of strain $\epsilon$ its closest point $P^+ \epsilon$ on $C^+$, Fig. 2. It can be shown that the effect of $P^+$ is simply to remove from $\epsilon$ its negative eigenvalue components. Thus, if $\epsilon^{(a)}$ and $d^{(a)}$, $a = 1,2,3$ denote the eigenvalues and eigenvectors of $\epsilon$, respectively, so that

$$\epsilon_{ij} = \sum_a \epsilon^{(a)} d_i^{(a)} d_j^{(a)},$$  \hspace{1cm} (3.8)

then the positive projection of $\epsilon$ is given by

$$(P^+ \epsilon)_{ij} = \sum_a \langle \epsilon^{(a)} \rangle d_i^{(a)} d_j^{(a)}$$  \hspace{1cm} (3.9)

where $\langle \cdot \rangle$ signifies the McAuley bracket, with its usual definition

$$\langle x \rangle = \frac{1}{2}(x + |x|).$$  \hspace{1cm} (3.10)
The solution to problem (3.7) can now be approximated as
\[ \epsilon = C^0 + P^+ (\Sigma^c + P^+ (\sigma)) \]
\[ = C^0 + P^+ (\Sigma^c : \sigma^+) \]  
(3.11)
where \( \sigma^+ = P^+ (\sigma) \) is the positive part of the stress tensor \( \sigma \). Stress-strain relations (3.11) can be expressed in a form consistent with (3.5) by making the identification
\[ C^c : \sigma = P^+ (\Sigma^c : \sigma^+) \]
(3.12)
and setting the effective added flexibility tensor due to the active microcracks equal to
\[ C^c = P^+ : \Sigma^c : P^+ \]  
(3.13)
Note that the effective flexibility due to microcracks in general depends on the state of stress. It is also interesting to note that the Gibbs potential (3.1) is explicitly given by
\[ G = \frac{1}{2} (\sigma : C^0 + \sigma^+ : \Sigma^c + \sigma^+) - A^c \]
(3.14)
It is clear from this expression that for purely compressive states of stress, i.e., for those states of stress for which \( \sigma^+ = 0 \), no microcracks become active and the flexibility tensor coincides with that of the uncracked material. Further insight into the nature of the closing process and its influence on the elastic response of the material may be gained by considering the following example.

**Example.** Consider a distribution of coplanar microcracks oriented along a common normal direction \( n \), Fig. 3. Then, for perfectly planar cracks it is clear that the total added flexibility due to the microcracks is of the form \( C^c = \mu n \otimes n \otimes n \otimes n \), for some scalar flexibility \( \mu \). Here, the symbol \( \otimes \) is used to signify tensor multiplication, i.e., in component form \( C_{ijkl}^c = \mu n_i n_j n_k n_l \). Fig. 3 depicts four loading cases which may be used to illustrate the dependence of the effective flexibility due to microcracks on the state of loading. Cases (a) and (c) in Fig. 3 represent two loading conditions for which \( \sigma^+ = 0 \). According to expression (3.14), therefore, such states of stress do not activate the microcracks, as was to be expected on intuitive grounds. Cases (b) and (d), by contrast, involve loading conditions which do have a nonzero positive projection, i.e., \( \sigma^+ \neq 0 \). Indeed, it is clear that \( \sigma^+ = \sigma \) and, hence, it follows from (3.14) that \( C^c = \Sigma^c \). However, in case (b), the orientation of the principal stresses in such that \( n \cdot \sigma^+ \cdot n = 0 \) and, thus, \( \epsilon^c = C^c : \sigma^+ = 0 \), i.e., the added flexibil-

![Fig. 2. Geometric representation of positive and negative cones in strain space and associated orthogonal projections. The added deformation due to opening of microcracks in splitting mode, or mode I, must lie in the positive cone \( C^+ \). Likewise, the added deformation arising from microcracks opening in compressive mode, or mode II, must be confined to the negative cone \( C^- \).](image)

![Fig. 3. Opening and closing logic for continuously distributed coplanar microcracks. The requirement that the opening of microcracks must result in added deformation lying in the positive cone acts as a unilateral constraint which precludes opening under loading conditions (a), (b) and (c).](image)
ity arising from the microcracks is not activated. Finally, case (d) gives \( \mathbf{n} \cdot \mathbf{\sigma}^+ \cdot \mathbf{n} \neq 0 \) and, consequently, \( \mathbf{\epsilon}^c \neq 0 \).

3.1.2. Cross effect

The closing conditions discussed above embody the intuitive notion that a perfectly planar crack opens only when acted upon by a tensile normal traction, Fig. 3(d). In turns out, however, that this is too restrictive an idealization of the state of microcracking of concrete, for which extended microcracks tend to follow a tortuous path, Fig. 4. This opens the possibility of cracks opening and thereby contributing to the overall flexibility of the material when acted upon by compressive stresses lying in the 'average' plane of the cracks, Fig. 4(b). This effect is henceforth referred to as the cross effect. Thus, microcracks are assumed to be capable of becoming active in two possible modes: mode I, or splitting mode, Fig. 4(a) and mode II or compressive mode, Fig. 4(b). Consideration of the cross effect is of crucial importance in modeling certain aspects of the material behavior of concrete. As mentioned above, it is well known that concrete under uniaxial compressive loading develops extended microcracks which run parallel to the axis of loading. Furthermore, it is observed that this process of damage results in a significant increase of the longitudinal flexibility of the material. This effect can only be explained if the possibility of cracks responding to compressive stresses acting in their average plane is taken into consideration.

The cross effect can be incorporated into the formulation by postulating an additive decomposition

\[
\overline{\mathbf{\epsilon}}^c = \overline{\mathbf{\epsilon}}^c_I + \overline{\mathbf{\epsilon}}^c_{II}
\]

of the total added flexibility tensor due to microcracks \( \overline{\mathbf{\epsilon}}^c \) into a term \( \overline{\mathbf{\epsilon}}^c_I \) due to the response of microcracks in mode I and a term \( \overline{\mathbf{\epsilon}}^c_{II} \) due to the response of microcracks in mode II. Thus, in the event that all of the microcracks in the material are active in both modes I and II, the deformation arising from the opening of the cracks would be given by

\[
\mathbf{\epsilon}^c \equiv \overline{\mathbf{\epsilon}}^c : \mathbf{\sigma} = \overline{\mathbf{\epsilon}}^c_I : \mathbf{\sigma} + \overline{\mathbf{\epsilon}}^c_{II} : \mathbf{\sigma} \equiv \mathbf{\epsilon}^c_I + \mathbf{\epsilon}^c_{II}
\]

where \( \mathbf{\epsilon}^c_I \) is the deformation due to mode I activation and \( \mathbf{\epsilon}^c_{II} \) the one due to mode II. The closing condition (3.6) may now be generalized to the requirement that

\[
\mathbf{\epsilon}^c_I \geq 0 \quad \text{and} \quad \mathbf{\epsilon}^c_{II} \leq 0
\]

i.e., that \( \mathbf{\epsilon}^c_I \) lie in the positive cone \( C^+ \) and \( \mathbf{\epsilon}^c_{II} \) in the negative cone \( C^- \), Fig. 2.

An entirely similar argument to the one put forward before immediately leads to the conclusion that the stress–strain relations compatible with (3.17) can be approximated as

\[
\mathbf{\epsilon} = \mathbf{C}^0 : \mathbf{\sigma} + \mathbf{P}^+ (\overline{\mathbf{C}}^c_I : \mathbf{P}^+ (\mathbf{\sigma})) + \mathbf{P}^- (\overline{\mathbf{C}}^c_{II} : \mathbf{P}^- (\mathbf{\sigma}))
\]

\[
= \mathbf{C}^0 : \mathbf{\sigma} + \mathbf{P}^+ (\overline{\mathbf{C}}^c_I : \mathbf{\sigma}^+) + \mathbf{P}^- (\overline{\mathbf{C}}^c_{II} : \mathbf{\sigma}^-)
\]

where \( \mathbf{P}^- = \mathbf{I} - \mathbf{P}^+ \) denotes the orthogonal projection onto the negative cone \( C^- \) and \( \mathbf{\sigma}^- = \mathbf{P}^- (\mathbf{\sigma}) \) is the negative part of the stress tensor \( \mathbf{\sigma} \). In other words, mode I deformation of the microcracks is activated by the positive part of the stress tensor \( \mathbf{\sigma}^+ \), whereas mode II is activated by the negative part \( \mathbf{\sigma}^- \).
The effective flexibility due to the active microcracks can be expressed as
\[ C^e \equiv P^+ : \overline{C}_1^e : P^+ + P^- : \overline{C}_2^e : P^-. \]  
(3.19)

and the Gibbs potential becomes
\[ G = \frac{1}{2} \left( \sigma : C^0 : \sigma + \sigma^+ : \overline{C}_1^e : \sigma^+ + \sigma^- : \overline{C}_2^e : \sigma^- \right) - A^e. \]  
(3.20)

It is interesting to note the duality of modes in which microcracks are now allowed to contribute to the overall flexibility of the material. This situation is further illustrated by the following example.

Example. Consider, as before, a distribution of coplanar microcracks oriented along a common normal direction \( n \), Fig. 5. By orientation of the crack it is now understood that of its average plane. The total added flexibility is now of the form \( \overline{C}^e = \mu_1 n \otimes n \otimes n + \mu_{11} m \otimes m \otimes m \), where the scalar flexibility coefficients \( \mu_1 \) and \( \mu_{11} \) refer to opening modes I and II, respectively, and \( m \) represent the unit vector along the direction of the average plane. Cases (a) and (c) in Fig. 5 represent two loading conditions for which \( \sigma^+ = 0 \) and \( \sigma^- = \sigma \). Thus, both states of stress may potentially activate mode II opening. However, the principal stress direction in case (c) is normal to the plane of the crack, i.e., \( m \cdot \sigma^- \cdot m = 0 \) and hence the cracks remain closed. By contrast, in case (a) stresses run parallel to the plane of the crack, which yields \( m \cdot \sigma^- \cdot m = 0 \) and hence the cracks are activated. On the other hand, cases (b) and (d) involve loading conditions for which \( \sigma^+ = \sigma \) and \( \sigma^- = 0 \). Thus, both states of stress could in principle activate mode I opening. However, in case (b) one has \( n \cdot \sigma^+ \cdot n = 0 \) and the cracks remain closed, whereas in case (d) \( n \cdot \sigma^+ \cdot n = 0 \) which activates the cracks in mode I.

3.1.3. Damage rules and damage criteria

The problem that remains, now, is to characterize the evolution of the damage processes and of the resulting added flexibility due to damage \( \overline{C}^e \). This task is greatly facilitated by some recently developed thermodynamic yield theories (Lubliner, 1975; 1978; 1980) which have significantly contributed to clarifying the connection between thermodynamics and rate independent inelastic processes. These theories rest upon the notion that the yield-like phenomena associated with rate independent behaviour is but a manifestation of the second law of thermodynamics. In plasticity, for example, once the flow rule is specified the yield criterion follows directly from the dissipation inequality.

Guided by these principles, a model for damage may be constructed by first postulating a rate independent damage rule of the following general form
\[ \overline{C}_1^e = \mu R_1(\sigma), \quad \overline{C}_2^e = \mu R_{11}(\sigma) \]  
(3.21)

where \( R_1(\sigma) \) and \( R_{11}(\sigma) \) are response functions of the material which determine the direction in which damage takes place and \( \mu \) is a scalar parameter which may be regarded as a measure of cumulative damage. Equations (3.21) may be combined according to (3.15) to yield
\[ \overline{C}^e = \mu \left[ R_1(\sigma) + R_{11}(\sigma) \right] = \mu R(\sigma) \]  
(3.22)

for the total added flexibility tensor due to microcracks, where one writes \( R(\sigma) = R_1(\sigma) + R_{11}(\sigma) \).

The irreversible character of damage necessitates
\[ \dot{\mu} = 0, \]  
(3.23)
i.e., that damage be an ever increasing function of time. The condition that \( \dot{\mu} > 0 \) signals that the damage mechanisms are active and damage to the material is being furthered. On the other hand, it follows from (3.21) that \( \dot{\mu} = 0 \) results in elastic behavior.

A criterion is needed now to ascertain under what conditions further damage occurs, i.e., \( \dot{\mu} > 0 \) and when the material remains elastic, i.e., \( \dot{\mu} = 0 \).

In classical plasticity, yield and loading criteria serve precisely this purpose. In the present context, a similar role is played by the damage and loading criteria to be defined next.

From the form of the Gibbs energy potential (3.14), it follows that the dissipation inequality is given by (Lubliner, 1972)

\[
d = \frac{1}{2} \sigma^* : C_1 \sigma^* + \frac{1}{2} \sigma^- : C_2 \sigma^- - A^* \geq 0
\]

(3.24)

where \( d \) is signifies the rate of dissipation. Inequality (3.24) stems directly from the second law of thermodynamics and must be satisfied by all processes undergone by the material. In (3.24), \( C_1 \) and \( C_2 \) represent the changes in mode I and II flexibilities due to the increase in damage, whereas \( A^* \) is the power dissipated in extending the microcracks. It is next assumed that the inelastic free energy \( A^* \) associated with microcrack formation may be expressed as a function of \( \mu \), i.e.,

\[
A^* = A^*(\mu).
\]

(3.25)

Furthermore, it proves convenient to define the quantity

\[
t(\mu)^2 = 2dA^*/d\mu.
\]

(3.26)

For reasons that will become apparent below, \( t \) may be termed the critical stress for the extension of damage. Making use of the damage rule (3.21) and definition (3.26), (3.24) becomes

\[
d = \left( \frac{1}{2} \sigma^* : R_1(\sigma) : \sigma^* + \frac{1}{2} \sigma^- : R_2(\sigma) : \sigma^- - \frac{1}{2} t(\mu)^2 \right) \mu \geq 0.
\]

(3.27)

But, since \( \dot{\mu} \) is constrained by the irreversibility assumption (3.23) to be greater of equal to zero, (3.27) necessitates

\[
\frac{1}{2} \sigma^* : R_1(\sigma) : \sigma^* + \frac{1}{2} \sigma^- : R_2(\sigma) : \sigma^- - \frac{1}{2} t(\mu)^2 \geq 0.
\]

(3.28)

This inequality can be further simplified by introducing the functions

\[
F_1(\sigma) = \frac{1}{2} \sigma^* : R_1(\sigma) : \sigma^* \\
F_2(\sigma) = \frac{1}{2} \sigma^- : R_2(\sigma) : \sigma^-
\]

(3.29)

Substituting these definitions into (3.28) one obtains

\[
\Phi(\sigma, \mu) = F(\sigma) - \frac{1}{2} t(\mu)^2 \geq 0
\]

(3.30)

where \( \Phi \) will be termed the damage function, and one writes

\[
F(\sigma) = F_1(\sigma) + F_2(\sigma).
\]

(3.31)

In other words, for further damage to occur, inequality (3.30) must be satisfied. Conversely, if inequality (3.30) is not satisfied, the material must behave elastically. Thus, (3.30) can be regarded as a damage criterion which signals the onset of damage. A classical argument reveals that, in the absence of viscosity, i.e., for rate independent material behavior, states of stress such that

\[
\Phi(\sigma, \mu) = F(\sigma) - \frac{1}{2} t(\mu)^2 > 0
\]

(3.32)

are unattainable, and the onset of damage is characterized by the criteria

\[
(\partial \Phi/\partial \sigma) : \sigma = (\partial F/\partial \sigma) : \sigma > 0.
\]

(3.33)

This latter condition plays an entirely analogous role to that played by the classical loading conditions in plasticity theory and will be likewise termed loading condition.

Inequalities (3.33) are amenable to a revealing geometric interpretation. The locus of points in stress space such that \( \Phi = 0 \) may be viewed as a damage surface enclosing an elastic domain within which the response of the material is elastic, i.e., does not result in further damage to the material. For damage to progress, two conditions must be simultaneously met: (i) The stress point must lie on the damage surface, and (ii) the stress increment must point outside the elastic domain.

For a single crack in an infinite medium, it is elementary to show that the above thermodynamic principles immediately yield the classical Griffith's
criterion for the extension of the crack. From this perspective, (3.21)–(3.33) would appear to constitute the natural generalization of Griffith's theory to the case of distributed damage. The proposed formulation defines a general phenomenological framework within which a model of damage is completely defined once two response functions of the material are specified: The damage directions \( R_1(\sigma) \) and \( R_{11}(\sigma) \) and the softening law \( t(\mu) \) giving the dependence of the critical stress \( t \) on the amount of cumulative damage \( \mu \). The task of identifying these response functions may be simplified by the following considerations.

### 3.1.4. Associated damage rules

A damage rule will be termed associated if the damage direction tensors are related to the damage function by

\[
R_1 = \frac{\partial^2 F_1}{\partial \sigma^+ \partial \sigma^+}, \quad R_{11} = \frac{\partial^2 F_{11}}{\partial \sigma^- \partial \sigma^-}.
\]

(M. Ortiz / Inelastic behavior of concrete

Making use of (3.29) and (3.30) it is readily seen that (3.34) necessitates

\[
F_1 = \frac{1}{2} \sigma^+ \cdot \frac{\partial^2 F_1}{\partial \sigma^+ \partial \sigma^+} : \sigma^+, \quad F_{11} = \frac{1}{2} \sigma^- \cdot \frac{\partial^2 F_{11}}{\partial \sigma^- \partial \sigma^-} : \sigma^-.
\]

But by Euler's theorem this is equivalent to requiring that \( F_1 \) and \( F_{11} \) be homogeneous functions of degree two of \( \sigma^+ \) and \( \sigma^- \) respectively.

The associativity assumption has the convenient effect of reducing the material identification problem to that of determining the scalar functions \( F_1 \) and \( F_{11} \), rather than the tensorial quantities \( R_1 \) and \( R_{11} \). As in classical plasticity, associativity also implies here normality in the following sense. Recalling (3.3) and making use of (3.4) and (3.21) it is found that

\[
\dot{\epsilon}^i = \dot{\mu} (R_1 : \sigma^+ + R_{11} : \sigma^-).
\]

Bringing now in relations (3.34) and invoking Euler's theorem for homogeneous functions (3.36) may be recast as

\[
\dot{\epsilon}^i = \dot{\mu} \left( \frac{\partial F_1}{\partial \sigma^+} + \frac{\partial F_{11}}{\partial \sigma^-} \right)
\]

which in view of (3.31) finally reduces to

\[
\dot{\epsilon}^i = \dot{\mu} \frac{\partial F}{\partial \sigma}.
\]

In other words, the inelastic part of the strain rate tensor points away from the damage surface and in the normal direction, Fig. 6.

### 3.1.5. Conjugacy relations

Further indications concerning the structure of the damage direction tensors \( R_1 \) and \( R_{11} \) may be derived from the classical conjugacy arguments of kinetic theory. It is seen from the dissipation inequality (3.27) that the thermodynamic fluxes conjugate to the variables \( \tilde{C}_1 \) and \( \tilde{C}_{11} \) are

\[
J_1 = \sigma^+ \otimes \sigma^+, \quad J_{11} = \sigma^- \otimes \sigma^-,
\]

respectively. It seems appropriate, therefore, to postulate that the damage processes are driven by the conjugate thermodynamic forces \( J_1 \) and \( J_{11} \) or, in the present rate-independent context, that the damage direction tensors depend on the state of stress through the conjugate thermodynamic fluxes. The simplest form of such dependence which is consistent with the associativity assumption is given by

\[
R_1 = \frac{J_1}{\text{Tr} J_1}, \quad R_{11} = c \frac{J_{11}}{\text{Tr} J_{11}}
\]

where \( \text{Tr} \) denotes the trace operation, i.e., \( \text{Tr} J_1 = (J_1)_{111} \) and \( c \) is a cross-effect coefficient which governs the extent of mode II damage. In particular, for \( c = 0 \) no cross-effect is present. Using relations (3.39), the resulting damage rule takes the form

\[
\dot{\tilde{C}}_1 = \dot{\mu} \frac{\sigma^+ \otimes \sigma^+}{\sigma^+ : \sigma^+}, \quad \dot{\tilde{C}}_{11} = \dot{\mu} \frac{\sigma^- \otimes \sigma^-}{\sigma^- : \sigma^-}
\]

and the corresponding damage function reads

\[
\Phi(\sigma, \mu) = \frac{1}{2} \sigma^+ : \sigma^+ + c \sigma^- : \sigma^- - \frac{1}{2} \dot{\mu}^2(\mu),
\]

i.e., \( F_1(\sigma) = \frac{1}{2} \sigma^+ : \sigma^+ \) and \( F_{11}(\sigma) = c \sigma^- : \sigma^- \). Note that \( F_1 \) and \( F_{11} \) are homogeneous functions of degree two of \( \sigma^+ \) and \( \sigma^- \), respectively, and thus the damage rule is associated.

The damage surface \( \Phi = 0 \) defined by (3.42) is shown in Fig. 6. It is seen that for a state of uniaxial tension \( \sigma \), the damage criterion reduces
to \( \sigma = t(\mu) \), which justifies the term 'critical stress' given to \( t \).

3.1.6. Determination of the softening law

For the damage model defined by (3.42), it is possible to determine the dependence of the critical stress \( t \) on the cumulative damage parameter \( \mu \) on the sole basis of the uniaxial tensile test. To see this, let us first note that for uniaxial tensile loading it follows from (3.41) that \( \hat{C}_{11} = 0 \), whereas \( \hat{C}_1 = \mu \mathbf{n} \otimes \mathbf{n} \otimes \mathbf{n} \otimes \mathbf{n} \), with \( \mathbf{n} \) being the loading direction. Thus, the total flexibility along the axis of loading is given by \( 1/E_0 + \mu \), where \( E_0 \) denotes the initial Young's modulus of the material. Hence, if we denote by \( \sigma(\epsilon) \) the stress–strain relation in uniaxial tension, it follows that

\[
\sigma = \frac{\epsilon}{1/E_0 + \mu} = \sigma(\epsilon).
\]

(3.43)

From the latter identity, it is possible to solve for \( \epsilon \) as a function of \( \mu \), say \( \epsilon = \epsilon(\mu) \), which together with the stress–strain relation finally yields

\[
t(\mu) = \sigma(\epsilon(\mu)).
\]

(3.44)

It is therefore concluded that the softening law is completely determined by the uniaxial tensile test. This situation parallels that encountered in isotropic plasticity where the hardening law follows also from the uniaxial test.

A convenient expression for the stress–strain relation of mortar in uniaxial tension is given by the equation

\[
\sigma = \frac{E_0 \epsilon}{1 + (E_0/E_\epsilon - 2)(\epsilon/\epsilon_\epsilon) + (\epsilon/\epsilon_\epsilon)^2}
\]

(3.45)

originally proposed by Saenz (1964) to describe the uniaxial compression stress–strain curve, Fig. 7(a). Here, \( \epsilon_\epsilon \), denotes the critical tensile strain and \( E_\epsilon = f_i/\epsilon_\epsilon \), is the secant stiffness at failure. Another plausible equation for the uniaxial tension stress–strain curve is furnished by

\[
\sigma = f_i(\epsilon/\epsilon_\epsilon)\exp(1 - \epsilon/\epsilon_\epsilon)
\]

(3.46)

due to Smith and Young (1955), also in the context of uniaxial compression, Fig. 7(a). The resulting softening laws are shown in Fig. 7(b). Numerical tests have indicated that the use of Smith and Young's curve yields results which are in better agreement with experiment than those based on Saenz's equation. Thus, the numerical results discussed in Section 4 have all been obtained using Smith and Young's curve.

3.1.7. Plastic microcracking

The damage model developed so far is based on the assumption that the material is perfectly brittle. The constitutive behavior so defined is such that no permanent strains are predicted to remain upon unloading of the material. Mortar, however, is known not to conform to this pattern (Adenaes, Gerstle and Ko, 1977). A sizable process zone, together with misfits on the surface of the cracks prevent them from closing completely and, thus, permanent strains develop. Similar effects have been noted in the context of metal fatigue (Suresh and Ritchie, 1983).

Clearly, the orientation of these permanent strains is determined by that of the extended microcracks. Therefore, it seems reasonable to assume that the rate of irrecoverable deformation associated with the opening of the cracks is coaxial.
with the total rate of inelastic deformation $\dot{\epsilon}^i$, (3.36). A simple model of plastic microcracking is obtained by further assuming that the rate of irrecoverable deformation is a constant fraction of $\dot{\epsilon}^i$, say, $\alpha \dot{\epsilon}^i$, where $\alpha$ is taken to be a material constant. Under these conditions, (3.3), (3.22) and (3.36) generalize to

$$\dot{\epsilon} = C : \dot{\sigma} + \dot{\epsilon}^i, \quad \dot{\epsilon}^i = \dot{\mu} (R_1 : \sigma^* + R_{11} : \sigma^*).$$

(3.47)

With this simple extension of the model, damage ranging from purely brittle to perfectly ductile can be readily accounted for. Thus, for $\alpha = 0$ the brittle limit discussed earlier is recovered. In particular, the rate of inelastic deformation $\dot{\epsilon}^i$ coincides with the rate of deformation $\dot{C} : \sigma$ due to the increase in damage, and the stress–strain curve unloads to the origin. For values of $\alpha$ between 0 and 1, the rate of inelastic deformation $\dot{\epsilon}^i$ is greater than $\dot{C} : \sigma$, the excess being the rate of increase of permanent deformations. Finally, for the limiting case of $\alpha = 1$, the elastic compliances remain constant and the material behaves plastically.

3.2. A simple plasticity model for aggregate

Aggregate is a granular material consisting of cohesionless particles that interact on contact. A distinct characteristic of this type of media is that of *positive dilatancy* whereby a granular mass under shearing experiences an increase in volume, following a small initial decrease. This effect was first noted by Reynolds (1885) and has been considered from various points of view in the past. Other outstanding aspects of granular material behavior, like the *noneoaxiality effect*, or lack of alignment between the principal directions of the applied stresses and the rate of deformation tensor, have also received the attention of numerous workers in the field (see, e.g., Nemat-Nasser, 1983; Cowin, 1978).

When imbedded in concrete, however, it is doubtful that all of the complexities in the behavior of aggregate as a granular material play a significant role. It seems therefore appropriate to use a simple model to describe the material behavior of aggregate. A criterion which is frequently used to characterize the failure of cohesionless soils is the Drucker–Prager (1952) failure criterion, which reads

$$\Phi(\sigma) = q - Mp,$$

(3.48)

where $p = \frac{1}{3} \sigma_{kk}$ is the hydrostatic pressure and $q = \sqrt{\frac{2}{3}s_{ij}s_{ij}}$, with $s_{ij} = \sigma_{ij} - p \delta_{ij}$, is the effective stress deviator. In view of the good agreement of the numerical tests shown below with experimental
data, the use of more elaborate material models for aggregate does not appear warranted.

A simple calculation readily shows that the internal friction coefficient $M$ is related to the internal friction angle $\phi$ by the expression

$$M = 6 \sin \phi / (3 - \sin \phi).$$  \hspace{1cm} (3.49)

It is well established, however, that in the presence of internal friction the normality rule ceases to be valid and plastic flow is nonassociated (Drucker, 1950; 1959; Mandel, 1966). This is in fact yet another distinct characteristic of granular media (Atkinson and Bransby, 1978) which manifests itself as the fact that the dilatancy coefficient $N$, defined as the ratio between volumetric and deviatoric plastic strain rates, does not in general coincide with the internal friction coefficient $M$. This motivates considering a flow potential of the type

$$\Psi(\sigma) = q - Np$$  \hspace{1cm} (3.50)

where in general $N \neq M$. The plastic strain rates are then given by

$$\dot{\varepsilon}^i = \lambda \frac{\partial \Psi(\sigma)}{\partial \sigma}$$  \hspace{1cm} (3.51)

and, since $\Psi \neq \Phi$, the plastic flow is nonassociated.

The flexibility of having independent material parameters governing failure and dilatation has been found of primary importance in the course of checking the predictions of the model against experimental data. In spite of the simplicity of the Drucker-Prager model, numerical tests would appear to indicate that it is accurate enough for modeling aggregate as a constituent of concrete. Nevertheless, one could conceivably use more elaborate models for granular media (see, e.g., Lade and Duncan, 1975; Nemat-Nasser and Shokooh, 1980) to improve this aspect of the theory.

3.3. Concrete as a mixture of mortar and aggregate

In the preceding sections, two constitutive models for mortar and aggregate viewed as separate materials have been outlined. The task to be faced now is to characterize the material behavior of concrete based on that of its constituents, mortar and aggregate. To this end, a possibility that immediately suggests itself is to use the theory of interacting continua, or theory of mixtures. In this respect, the present approach sharply departs from most theories proposed up to the date which regard concrete as a simple or single-phase material. Far from being an unnecessary detour, the use of the theory of interacting continua as a basis for devising material laws for concrete offers significant advantages. Foremost among these is the notion of phase stresses, i.e., that the externally applied stresses $\sigma$ distribute unevenly between mortar and aggregate. These phase stresses jointly equilibrate the external ones, but are in general vastly different from each other.

Thus, for instance, the numerical tests presented below point to the fact that, when concrete is subjected to a uniaxial compression, high splitting stresses normal to the axis of loading develop in mortar. This explains the characteristic splitting failure mode commonly observed in uniaxially compressed samples (Nelissen, 1972; Wastil, 1979) and illustrates the fact that in general the inelastic processes that take place in mortar and aggregate are driven by different states of stress. A further effect whose explanation and quantification is greatly facilitated by the use of mixture theory is that of the unloading hysteretic loops that develop during processes of cyclic loading (Karsan and Jirsa, 1969; Sinha, Gerstle and Tulin, 1964; Spooner and Dougill, 1975). Thus, the theory shows that, as plastic flow ensues, strong residual stresses develop in both mortar and aggregate. In the latter, these residual stresses suffice to cause reverse yielding upon unloading of the material, thus giving rise to hysteretic loops.

In conclusion, the potential pay-off of the use of mixture theory lies in that simple models for the phases suffice to capture the complexities of the overall material behavior. This would also appear to be a further indication of the fundamental role played by the composite nature of concrete in shaping its constitutive behavior. It seems therefore warranted to study in depth the consequences of this aspect of the material as a first step towards devising adequate material laws for concrete. It would appear that similar conclusions apply also
to *geomaterials* such as sandstones in which silica grains are cemented together by lime, iron oxides and clay-like substances.

A detailed study of concrete as a mixture has been presented elsewhere by the author (Ortiz and Popov, 1982a). Following an approach due to Green and Naghdi (1965) it was shown that the *phase stresses* $\sigma_1$ and $\sigma_2$ acting in mortar and aggregate, respectively, are related to the externally applied stresses through the expression

$$\sigma = \alpha_1 \sigma_1 + \alpha_2 \sigma_2$$

(3.52)

where $\alpha_1$ and $\alpha_2$ are the corresponding volumetric fractions. These phase stresses are to be understood in the sense of the theory of interacting continua. In particular, $\sigma_2$ does *not* represent the stresses within the aggregate particles but rather it accounts for the contact forces that develop between them. In fact, the notion of aggregate particle as a finite domain is lost in the context of mixture theory. Thus, it is assumed that an arbitrarily small volume of concrete contains both mortar and aggregate in fixed proportions. This is a plausible assumption provided that the only processes which are considered are macroscopic, i.e., have a characteristic length which is large compared with the size of the aggregate particles.

On the other hand, it follows from the principle of angular momentum balance that the applied stresses $\sigma$ are symmetric. This, however, does not imply that the phase stresses $\sigma_1$ and $\sigma_2$ are themselves symmetric. In fact, one frequently finds in liquid mixtures that the phases interact through stress couples. For the purpose at hand, however, it is assumed that the phase stresses in mortar and aggregate are both symmetric.

The absence of diffusion between mortar and aggregate results in the following compatibility conditions

$$\epsilon_1 = \epsilon_2 = \epsilon$$

(3.53)

where $\epsilon_1$, $\epsilon_2$, and $\epsilon$ are the strain tensors of mortar, aggregate and concrete, respectively. As in the case of the phase stresses, it should be strongly emphasized here that the strain tensors in (3.53) refer to the *macroscopic deformation of the phases* and are *not* in any way intended as measures of the deformation processes that take place at the microstructural level. In particular, $\epsilon_2$ does *not* refer to the deformation of the aggregate particles, but it rather describes the deformations undergone by volumes of aggregate containing many aggregate particles. Thus, the compatibility relation (3.53) takes a completely different meaning and serves a different purpose than the similar one which is postulated in Taylor’s method for composite materials (Taylor, 1938). In this latter case, compatibility of deformations between the matrix and the inclusions is postulated at the microstructural level. This is a rather restrictive assumption that has been relaxed in various ways in subsequent refinements of the theory (see, e.g., Kröner, 1958; Hill, 1965; Budiansky, 1965; Mura, 1982). In the mixture model used here, compatibility of deformations is enforced at the *macroscopic level* and is a direct consequence of the absence of diffusion between the phases.

The composition rule for phase stresses (3.52) and the compatibility condition for phase deformations (3.53) is all which is needed to derive the overall stress–strain relations for concrete. To this end, let us first recall that the material models proposed in preceding sections for mortar and aggregate lead to stress–strain relations of the type

$$\epsilon_1 = C_1 : \sigma_1 + \epsilon_1^p,$$

$$\epsilon_2 = C_2 : \sigma_2 + \epsilon_2^p$$

(3.54)

where $C_1$ and $C_2$ are the instantaneous elastic flexibility compliances of mortar and aggregate, respectively, and $\epsilon_1^p$ and $\epsilon_2^p$ the corresponding plastic strains. As discussed above, $\epsilon_1$ evolves in time as a result of damage, and the plastic deformations $\epsilon_1^p$ relate to plastic opening of the microcracks. By contrast, the elastic compliances $C_2$ may be assumed to remain constant and $\epsilon_2$ is the result of the slip-type plastic flow of aggregate.

Substituting (3.54) into (3.52) and making use of the compatibility relations (3.53), it is found that

$$\sigma = \alpha_1 \sigma_1 + \alpha_2 \sigma_2 = \alpha_1 D_1 : (\epsilon - \epsilon_1^p) + \alpha_2 D_2 : (\epsilon - \epsilon_2^p)$$

$$= D : (\epsilon - \epsilon^p)$$

(3.55)

where $D_1 = C_1^{-1}$ and $D_2 = C_2^{-1}$ are the phase stiff-
ness compliances and one writes

\[ D = \alpha_1 D_1 + \alpha_2 D_2, \]

\[ \epsilon^p = C : (\alpha_1 D_1 : \epsilon_1^p + \alpha_2 D_2 : \epsilon_2^p), \quad C = D^{-1}. \quad (3.56) \]

for the stiffness tensor and plastic deformation of concrete. Solving for \( \epsilon \) in (3.55) and substituting into (3.54) yields the following relation between the phase stresses and the externally applied ones

\[ \sigma_1 = B_1 : \sigma + \rho_1, \]

\[ \sigma_2 = B_2 : \sigma + \rho_2 \quad (3.57) \]

where the influence tensors \( B_1 \) and \( B_2 \) and the residual stresses \( \rho_1 \) and \( \rho_2 \) are defined as

\[ B_1 = D_1 : C, \quad \rho_1 = D_1 : (\epsilon^p - \epsilon_1^p), \]

\[ B_2 = D_2 : C, \quad \rho_2 = D_2 : (\epsilon^p - \epsilon_2^p). \quad (3.58) \]

Some aspects of (3.57) merit further comment. Thus, it is interesting to note that the phase stresses are the sum of two terms: a load induced term and residual stresses. The former arises as a direct consequence of the stresses \( \sigma \) that are applied to concrete. The influence tensors determine how these stresses distribute between mortar and aggregate. The value of the influence tensors in turn determined by the relative stiffnesses of mortar, aggregate and concrete.

The residual stresses are the ones that would remain in the phases if the external loads were removed. From their definition, it is apparent that the residual stresses are the result of the plastic flow of the phases. The influence tensors and residual stresses satisfy the following identities

\[ \alpha_1 B_1 + \alpha_2 B_2 = I, \]

\[ \alpha_1 \rho_1 + \alpha_2 \rho_2 = 0 \quad (3.59) \]

which follow directly from their definition. Equation (3.59a) guarantees the satisfaction of the composition rule (3.52) for any choice of external stresses \( \sigma \). On the other hand, (3.59b) indicates that the residual stresses \( \rho_1 \) and \( \rho_2 \) are in equilibrium with zero applied stresses. Similar identities are satisfied by influence tensors derived by means of self-consistent methods (see, e.g., Hill, 1965).

The mixture-theoretical picture outline above sheds considerable light on the inner workings of the inelastic response of concrete, which appear as a subtle interaction between mortar and aggregate. Thus, for instance, the degradation of the elastic properties of mortar has the effect of altering the influence tensors \( B_1 \) and \( B_2 \). This determines a redistribution of stresses between mortar and aggregate which frequently contributes to the stability of the mixture. Thus, for instance, the numerical results presented below indicate that in the uniaxial compression test this redistribution of stresses tends to relieve the splitting stresses that develop in mortar and which cause the extension of microcracks. This provides a plausible explanation for the observed fact that microcrack extension is far more stable in compression than in tension (Gluklich, 1963; Newman, 1968). This effect endows concrete with a much desirable ductility in compression which may be credited in part for its success as an engineering material.

Another intriguing aspect of the inelastic behaviour of concrete which is amenable to a plausible explanation within the context of mixture theory is the development of hysteretic unloading loops. It follows from (3.57) that upon removal of the applied stresses, i.e., upon setting \( \sigma = 0 \), residual stresses \( \rho_1 \) and \( \rho_2 \) remain in mortar and aggregate. Numerical experiments indicate that the latter are frequently strong enough to cause reverse failure in aggregate well before the applied stresses are completely removed.

3.4. Summary of constitutive relations

Before proceeding on to discussing numerical results, a summary of the constitutive relations utilized in the computations is given next. The material parameters involved in the model are:

- Concrete: Volumetric fractions of mortar and aggregate, \( \alpha_1, \alpha_2 \).
- Mortar: Young's modulus and Poisson's ratio of uncracked mortar, tensile failure stress and strain, \( f_t, \epsilon_t \); cross-effect coefficient \( c \) (\( \approx 0.04 \)), and plastic cracking coefficient \( a ( \approx 0.4 \).
- Aggregate: Young's modulus and Poisson's ratio; internal friction coefficient \( M \), and dilatancy coefficient \( N \).

All of these parameters are measurable, physically
meaningful quantities which can be readily assigned numerical values.

The proposed constitutive equations for concrete are strongly path dependent and their integration has to be carried out incrementally. Let us assume that the state variables \((\epsilon, \sigma_1, \mu_1, C_1, C_1^\vee, \epsilon_1', \sigma_2, \mu_2, \epsilon_2')\) are known at some given time, where \(\epsilon\) is the strain tensor of concrete, \((\sigma_1, \mu_1, C_1, C_1^\vee, \epsilon_1')\) denote the stresses, cumulative damage parameter, added flexibilities due to damage and inelastic strains of mortar and \((\sigma_2, \mu_2, \epsilon_2')\) are the stresses, cumulative plastic parameter and inelastic strains of aggregate. Then, a subsequent integration step involves the following sequence of operations:

(a) Computation of the elastic compliances of mortar. To this end, let \((\sigma_1^{(n)}\) and \((d_1^{(n)})\), \(n = 1, 2, 3\), denote the principal stresses and directions of \(\sigma_1\). The tensile part of \(\sigma_1\) and the corresponding positive projection are defined as

\[
\sigma_1^+ = \sum_{\alpha} \sigma_1^{(n)} d_1^{(n)} \otimes d_1^{(n)},
\]

\[
P_1^+ = \sum_{\alpha} d_1^{(n)} \otimes d_1^{(n)} \otimes d_1^{(n)} \otimes d_1^{(n)}
\]

where the sums extend over the tensile principal directions, i.e., to those principal directions for which \(\sigma_1^{(n)} > 0\). On the other hand, the compressive part of \(\sigma_1\) and the corresponding negative projection are given by

\[
\sigma_1^- = \sigma - \sigma_1^+,
\]

\[
P_1^- = I - P_1^+
\]

where the identity tensor reads \(I_{ij} = \delta_{ij}\). The effective added flexibility due to damage then follows from the expression.

\[
C_1 = P_1^+ : \bar{C}_1 : P_1^+ + P_1^- : \bar{C}_1^\vee : P_1^-
\]

and the total elastic compliances of mortar are given by

\[
C_1 = C_1^\vee + C_1^\vee, \quad D_1 = C_1^{-1}
\]

where \(C_1\) is the instantaneous elastic flexibility tensor of mortar and \(D_1\) the corresponding stiffness tensor.

(b) Computation of the tangent stiffness of mortar. To compute the tangent stiffness of mortar, let us recall that the damage rule, elastic relations and consistency condition may be chosen to be

\[
\ddot{C}_1 = (1 - \alpha) \mu_1 \frac{\sigma_1^+ \otimes \sigma_1^+}{\sigma_1^+ : \sigma_1^+},
\]

\[
\ddot{C}_1^\vee = (1 - \alpha) \mu_1 c \frac{\sigma_1^- \otimes \sigma_1^-}{\sigma_1^- : \sigma_1^-},
\]

\[
\dot{\epsilon}_1 = C_1 : \dot{\sigma}_1 + \epsilon_1', \quad \dot{\epsilon}_1 = \mu_1 s_1,
\]

\[
\Phi_1 = s_1 : \dot{\sigma}_1 - t_1(\mu_1) t_1'(\mu_1) \mu_1 = 0.
\]

where \(\epsilon_1\) is the strain tensor of mortar, \(\Phi_1\) the damage function \(t_1(\mu_1)\) the critical stress and \(s_1 = \sigma_1^- + c \sigma_1^-\) is an effective stress for damage. Furthermore, assuming that the tensile stress–strain curve is given by the Smith and Young function (3.46), the dependence of the critical stress \(t_1\) on \(\mu_1\) takes the form

\[
t_1(\mu_1) = f_1 e \frac{\log(1 + E_1''(\mu_1))}{1 + E_1''(\mu_1)}
\]

where \(E_1''\) is the Young’s modulus of uncracked mortar and \(e = 2.71828\). Combining equations (3.64) one readily finds

\[
\dot{\mu}_1 = \frac{s_1 : D_1 : \dot{\epsilon}_1}{s_1 : D_1 : s_1 + t_1 t_1'},
\]

\[
\dot{\sigma}_1 = \left[ D_1 - \left( \frac{D_1 : s_1}{s_1} \otimes \left( D_1 : s_1 \right) \right) \right] : \dot{\epsilon}_1 \equiv D_1^{(T)} : \dot{\epsilon}_1.
\]

where \(D_1^{(T)}\) signifies the tangent stiffness tensor of mortar.

(c) Computation of the tangent stiffness of aggregate. Assuming the validity of the Drucker–Prager plasticity model for aggregate, it follows that the flow rule, elastic relations and consistency condition take the form

\[
\dot{\epsilon}_2 = \dot{\mu}_2 \left( \frac{3 s_2}{2 q_2} - \frac{N}{3} I \right) \equiv \dot{\mu}_2 \eta_2,
\]

\[
\dot{\epsilon}_2 = C_2 : \dot{\sigma}_2 + \epsilon_2',
\]

\[
\Phi_2 = \left( \frac{3 s_2}{2 q_2} - \frac{M}{3} I \right) : \dot{\sigma}_2 \equiv \psi_2 : \dot{\sigma}_2 = 0.
\]

where \(\epsilon_2\) is the strain tensor of aggregate, \(s_2\) is the deviatoric part of \(\sigma_2\), \(q_2 = \sqrt{3/2} s_2 : s_2\) and \(\eta_2\) and \(\psi_2\) denote the direction of plastic flow and the normal
to the yield surface, respectively. As in the case of aggregate, combining equations (3.67) leads to

$$\bar{\mu}_2 = \frac{\eta_2 : D_2 : \bar{\epsilon}_2}{\eta_2 : D_2 : \nu_2},$$

$$\bar{\sigma}_2 = \left[ D_2 - \frac{(D_2 : \nu_2) \otimes (D_2 : \eta_2)}{\eta_2 : D_2 : \nu_2} \right] \cdot \bar{\epsilon}_2 = D_2^{(T)} : \bar{\epsilon}_2,$$

(3.68)

where $D_2^{(T)}$ signifies the tangent stiffness tensor of aggregate.

(d) Computation of tangent stiffness of concrete. The mixture theory relations (3.52) and (3.53) can be formulated in rate form as

$$\dot{\sigma} = \dot{\sigma}_1 + \alpha_2 \dot{\sigma}_2, \quad \dot{\epsilon}_1 = \dot{\epsilon}_2 = \dot{\epsilon}.$$  

(3.69)

Substituting (3.66) and (3.68) into (3.69) one finds

$$\dot{\sigma} = D^{(T)} : \dot{\epsilon}, \quad D^{(T)} = \alpha_1 D_1^{(T)} + \alpha_2 D_2^{(T)}.$$

(3.70)

where $D^{(T)}$ is the sought tangent stiffness for concrete.

(e) Update of state variables. Rate constitutive equations (3.70) express an incremental relation between the overall stresses and strains of concrete. For the integration process to be well-defined, the nature of the applied loading and the kinematic constraints acting on the sample need to be specified. In a finite element analysis, for instance, rate constitutive equations (3.70) are combined with a weak form of rate of momentum balance to obtain a linearized system of equations which can be solved for the incremental displacements. From these, the incremental strains can be computed. Once the incremental strains are known, the remaining state variables can be directly updated from equations (3.69b), (3.64), (3.67) and (3.69a).

4. Numerical tests

A general theory of the inelasticity of concrete has been outlined in the preceding section which would appear to exhibit considerable potential as a basis for a systematic explanation of the observed material behavior. To completely fulfill the objectives put forward at the introduction, it remains to check the ability of the theory to reproduce within the bounds of experimental error the available data and the suitability of the model for use in computation. In this section, the former issue is addressed by comparing some tests of the model to experimental data.

A word of caution concerning this and similar exercises seems warranted. Some authors have solely emphasized good fit to experimental records as a means of testing the validity of their models. However, the scatter in the available data is so large that such emphasis seems hardly warranted. The outcome of the tests is known to be extremely sensitive to the experimental procedure. Examples in this respect are numerous and conclusive. In a review paper by Wile (1968), for instance, the values of the consolidation factor (ratio of biaxial to uniaxial strength) determined experimentally by several authors were compared. Such values ranged from 0.90 to as much as 3.43 in certain cases. The high values of the consolidation factor measured by some authors are attributable to excessive friction of the specimen with the bearing platens. Modern experimental procedures have partly corrected this deficiency, but considerable scatter remains (Taylor and Patel, 1974; Chen and Chen, 1975; Adenaes, Gerstle and Ko, 1977).

A further example is furnished by two supposedly identical series of experiments conducted on one concrete, the first at the Technical University of Munich using brush bearing platens (Lins and Aschl, 1976), the other at the Federal Material Testing Laboratory in Berlin using flexible platens (Schickert and Winkler, 1977). Octahedral volumetric stress–strain curves were produced for various principal stress ratios. Although qualitatively similar, the measurements obtained by the two methods are so far off from each other that doubts concerning the validity of the data could be legitimately voiced.

In view of the present shortcomings of state-of-the-art experimental concrete research, it would appear that the significance to be attributed to the available data is mainly one of a qualitative description of the main features of the material behavior. Thus, given the extent of experimental error, insisting in curve fitting exercises would seem unwarranted. Instead, a good qualitative agreement with the data should be sought, in
conjunction with other equally desirable attributes of a material model.

One of such desirable feature is that the material constants involved in the model have a clear physical interpretation, so that values corresponding to particular materials can be readily determined. The parameters involved in the present model directly relate to the uniaxial tension stress-strain curve and the internal friction and dilatancy properties of the material, and their determination by a user minimally acquainted with the properties of concrete is straightforward. To further illustrate this point, in all of the numerical experiments shown next average values have been assigned to all parameters without making any attempt whatsoever at optimizing the fit to experimental records. This feature of the model greatly enhances its applicability in practical situations and is in sharp contrast to other currently proposed phenomenological models which involve material parameters that are devoid of physical meaning and whose determination thus requires elaborate optimization schemes.

4.1. Uniaxial tension and compression tests

The first example is concerned with the uniaxial compression and tension tests. Fig. 8. The results obtained from the model are seen to be in good agreement with experimental data from Bresler and Bertero (1979). The model correctly predicts a ratio of tensile to compressive strengths of about 0.1, in keeping with experimental observations (Jones and Kaplan, 1957). Furthermore, the ratio of the strains at the tensile and compressive peaks is predicted to be in the range 0.20–0.25, also in agreement with experimental measurements (Hughes and Chapman, 1966).

As damage accumulates, the flexibility tensor of concrete is seen to take the characteristic orthotropic profile

\[
C = \begin{pmatrix}
1/E_t & -\nu/E_t & 0 & 0 & 0 \\
-\nu/E_t & 1/E_t & 0 & 0 & 0 \\
-\nu/E_t & -\nu/E_t & 1/E_t & 0 & 0 \\
0 & 0 & 0 & 1/G_t & 0 \\
0 & 0 & 0 & 0 & 1/G_t
\end{pmatrix}
\]

(4.1)

as expected from the microcrack pattern which is observed to develop during the uniaxial compression test. In (4.1) the coordinate direction \(x_3\) is taken to coincide with the axis of loading. Plots of the various orthotropic moduli versus axial deformation are shown in Figs. 9(a) and 9(b). The marked elastic degradation that ensues as a consequence of damage is apparent from these plots. A curious phenomenon is observed in connection with the transverse Poisson's ratio \(\nu\), Fig. 9(b). Coincident with the onset of unstable behavior, \(\nu\) is seen to become negative. This bears on the fact that a small superimposed lateral compression would tend to close the longitudinal extended cracks that are present in the material, thus resulting in a decrease in deformation along the normal direction, instead of an increase.

It is regrettable that none of the available experimental data can be readily compared to these results. Almost invariably, the data collected heretofore concerning the evolution of the 'elastic' moduli has been based on secant, isotropic definitions (Hsu et al., 1963; Gardner, 1969; Mills and Zimmerman, 1970; 1971; Linse, 1973; Palaniswamy and Shah, 1974). It should be noted, however, that in spite of the significance attributed to these measurements, they do not relate directly to
the elastic response of the material, in the sense of the response that would be obtained if further inelasticity of the material were prevented. Rather on the contrary, secant moduli involve the whole response of the material, instead of its current elastic properties. Furthermore, in view of the strongly anisotropic character which the elasticity of concrete is seen to take, notions inspired on isotropic elasticity such as that of a bulk or shear modulus would appear to lose their applicability. In sum, this is one area which could certainly benefit from further experimental work aimed at measuring the evolution of the elastic response of concrete without interference from the inelastic processes.

Finally, Fig. 10 depicts the values of the phase stresses that develop in mortar and aggregate normalized by the applied axial stress $\sigma$, as a function of the axial deformation. In this plot, $\sigma_i^{(1)}$ and $\sigma_i^{(2)}$ signify the axial stresses in mortar and aggregate, respectively, and $\sigma_j^{(1)}$ and $\sigma_j^{(2)}$ the corresponding lateral stresses. Some aspects of these curves are noteworthy. Thus, for instance, splitting stresses $\sigma_j^{(1)}$ large enough to cause significant damage in mortar are observed to develop. By the cross effect discussed earlier, this transverse damage results also in moderate softening in the axial direction.

Eventually, this softening becomes substantial enough to induce a descending branch the the stress–strain curve, Fig. 8, although at a much later stage than in the tensile test case. It is interesting to note that these splitting stresses are due to the tendency of aggregate to be squeezed sideways. Thus, as the load bearing capacity or mortar
in the lateral direction diminishes, the fraction of load taken by aggregate must in turn decrease so as to mitigate this 'squeezing out' effect. This accounts for the unloading which is observed to take place in aggregate as deformation progresses, Fig. 10. As a result, most of the applied load ends up being taken directly by mortar, as implied by the upward tendency of the evolution of $\sigma^2_{11}$ in Fig. 10.

In conclusion, further to obtaining a good agreement with experimental data the proposed model provides some revealing insights as to internal mechanisms underlying the material response of concrete. This dual role played by the model would appear to be one of its strongest points.

4.2. Cyclic loading

Figure 11(a) shows the uniaxial cyclic response of the material predicted by the model. Fig. 11(b), on the other hand, shows corresponding experimental results obtained by Karsan and Jirsa (1969). As may be seen, the model accurately captures the relevant features of the response. Thus, for instance, a drop of about 16% in the compressive strength of the material with respect to its monotonic strength is correctly predicted. On the other hand, the computed hysteretic unloading loops closely follow after the observed pattern.

The origin of the unloading hysteretic loops may be traced back to the development of residual stresses in aggregate. As the specimen is loaded in compression, the tendency of aggregate is to flow sideways, restrained only by mortar. This interaction results in high residual stresses $\rho_1$ and $\rho_2$ in both phases. In particular, the residual stresses $\rho_2$ that appear in aggregate tend to oppose its lateral flow. Upon unloading of the material, these residual stresses become dominant and cause reverse yielding in aggregate, thus resulting in unloading hysteretic loops.

4.3. Biaxial test data

Figs. 12(a) and 12(b) show a comparison between the computed biaxial failure envelope and test data by Liu, Nilson and Slate (1972a), Adenaes, Gerstle and Ko (1977) and Kupfer, Hilsdorf and Rusch (1969). The model correctly captures the vastly differing behavior of concrete in biaxial tension and compression. The predicted biaxial consolidation factor is of the order of 1.3, which falls within the range of experimental observations. Overall, the computed failure envelope is seen to agree with the data within the bounds of experimental error.

A further satisfactory aspect of the model is that it succeeds in correctly predicting the failure
mode of the specimen under the various load combinations. Thus, under biaxial compression the model predicts a splitting failure mode, with extended cracks contained within the plane of loading. For uniaxial compression a double family of longitudinal cracks is predicted, whereas for uniaxial tension extended cracking takes place normal to the axis of loading. All of these results are in keeping with experimental observations (Nelissen, 1972).

Figs. 13(a) and 13(b) show the computed volumetric response for biaxial loading and the experimental measurements by Gerstle and Linse (1978). It should be noted that this particular test has
proven extremely sensitive to the method of testing. The use of brush bearing platsens (Linse and Aschl, 1976), or flexible platsens (Schickert and Winkler, 1977) results in significant quantitative discrepancies in the data. All of the experimental measurements, nevertheless, do exhibit common qualitative features which are captured by the model. Thus, it is generally observed that under increasing compression the material first contracts and eventually dilates owing to extensive microcracking. The onset of dilatation occurs immediately preceding failure in uniaxial compression and is retarded by lateral confinement, as predicted by the model.

4.4. Triaxial test data

The final set of numerical examples is concerned with the triaxial behavior of concrete. The tests by Balmer (1949) demonstrate the strongly stabilizing effect of confining pressure on the stress–strain response of concrete. Fig. 14(b). This stabilization effect is also present in the numerical results shown in Fig. 14(a), which exhibit little or no unstable behavior at high enough confinement. The origin of this added stability may be found in the restraining effect exerted by the confining pressure on the extension of microcracks. Under these circumstances, faulting becomes the predominant mode of failure predicted by the model, which is in keeping with experimental observations (Balmer, 1949; Palaniswamy and Shah, 1974). Finally, the increase of ultimate axial strength computed from the model as a function of the confining pressure is also in good agreement with the experimental data.

5. Concluding remarks

It has been shown in this paper how the body of theory commonly referred to as theory of interacting continua or theory of mixtures can be brought to bear on the constitutive behavior of concrete. This is a novel application of these ideas, with the advantage over other models that the inelastic processes in mortar and aggregate are driven by phases stresses which jointly equilibrate the applied ones but which may differ vastly from each other. This is seen to lie at the foundation of some aspects of the phenomenology of concrete which had heretofore defied satisfactory explanation, such as the characteristic splitting modes under biaxial compression and the unloading hysteretic loops under cyclic loading. The other major constituent of the proposed theory is a rate independent model of damage, which can accommodate fully anisotropic elastic degradation in a par-

Fig. 14. Triaxial stress-strain curves, computed (a) and experimental (b). Note the hardening and stabilizing effect of confinement.
particularly convenient manner. Numerical tests demonstrate the good agreement of the predictions of the theory with experimental data.

The applicability of the proposed model can be enhanced by extending it in the following directions:
- Incorporation of rate sensitivity effects.
- Incorporation of creep effects.

Rate sensitivity effects can be adequately accounted for by extending the rate independent model outlined above into the viscoplastic range. The procedure to be followed for such an extension is presently well understood (Lubliner, 1972). On the other hand, the body of experimental data concerning the effect of strain rate on the response of concrete is extensive and strongly suggests the correct form of the response functions involved. Rheological effects such as creep can be incorporated into the formulation through the notion of thermal activation (Krausz and Eyring, 1975). Eyring's theory of thermal activation offers ample possibilities in terms of modeling the nonlinearities that are observed in the rheology of concrete. The form of temperature dependence which is peculiar to thermal activation theories has been found to be in agreement with the available data (Bazant and Panula, 1979; Bazant, Tsubaki and Celep, 1983). The key response functional to be determined is the precise form of the activation energy for both damage and plastic flow. As in the case of rate effects, the experimental evidence concerning creep in concrete is extensive and should facilitate a straightforward characterization of the activation energy function. These and related issues will be the subject of a forthcoming publication.

References


Bresler, B. and V.V. Bertero (1979), "Influence of high strain rate and cyclic loading on behavior of unconfined and confined concrete in compression", Report No. EERC 75-16, Dept. of Civil Engineering, University of California, Berkeley.


