DISTORTIONAL HARDENING RULES FOR METAL PLASTICITY

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ABSTRACT: A brief overview of the available experimental data regarding distortional hardening of metals is first presented. This material is subsequently used to motivate the need for accurate distortional hardening rules in computation. A general expression for the yield surface of a plastic material is proposed that includes the isotropic-kinematic von Mises model as a particular case and that can be systematically used to incorporate distortional hardening features into the material modeling in a simple manner. This expression is complemented with suitable rate equations for the parameters involved. The proposed model is particularly convenient for computer implementation.

INTRODUCTION

The term inviscid plasticity denotes a broad concept encompassing a variety of phenomena that are observed in inelastic materials such as metals (1). In the historical origins of the subject, however, plasticity had a more limited and concrete scope. Basically, it was an attempt to systematize and formalize simple experimental observations such as the fact that metals can only sustain a limited level of stress and exhibit irreversible or plastic deformations. Thus, early experimental work (8,9,25) was directed towards the determination of elastic regions and flow laws. Advances in the theory of plasticity soon motivated more elaborate experimental work in which the determination or verification of hardening rules attracted considerable attention (10,13,14,16,19,21,24).

This wealth of experimental information has motivated extensive theoretical work concerning the interpretation and modeling of the observed material behavior. Following early attempts in this direction (7,23,29), very refined phenomenological models have been proposed that allow accurate predictions under arbitrary loading (5,6,11,18,20,22). Most of these models are based on multiple-surface formalisms and elaborate stress-strain laws which involve considerable computational effort. It is not surprising, therefore, that in practical applications the possibility of such refinements is frequently ignored and simple single-surface, bilinear-hardening models are employed. Despite their simplicity, these models do meet the two aforementioned basic objectives of plasticity, namely, to limit stresses to some bounded domain, or elastic region, and to allow for plastic flow of the material. On the other hand, single-surface models can be viewed as a systematic approximation to the real material behavior in the following sense. Based on examination of numerous experimental records, it has been shown in (4) that uniaxial stress-strain diagrams for metals approach asymptotically a bounding line such

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Note.—Discussion open until January 1, 1984. To extend the closing date one month, a written request must be filed with the ASCE Manager of Technical and Professional Publications. The manuscript for this paper was submitted for review and possible publication on May 12, 1982. This paper is part of the Journal of Engineering Mechanics, Vol. 109, No. 4, August, 1983. ©ASCE, ISSN 0733-9399/83/0004-1042/$01.00. Paper No. 18157.
as BC in Fig. 1. Thus, a suitable approximation to the stress-strain diagram is furnished by the bilinear diagram consisting of the initial elastic tangent AB and the asymptote or bounding line BC, Fig. 1. From this point of view, the yield point of the material can be taken to be B, i.e., the "rounded knee" transition from elastic to plastic behavior can be neglected. This reasoning leads to the familiar bilinear representation of material behavior. In adopting this approach, the overall "allowable region" for stresses and the flow of the material are reasonably well approximated.

For multiaxial states of stress, the locus of the points B for different directions of loading defines a surface in stress space which is commonly termed "limit surface" (15). This definition of yield is known as the "back-extrapolation" method. Another possible definition of yield corresponds to the elastic or proportionality limit, point E in Fig. 1. In the multiaxial case this determines a yield surface that is always contained in the limit surface defined above. Note that the yield surface contains all the states of stress for which the material is elastic whereas the limit surface determines a region to which attainable states of stress are confined. On the other hand, the yield surface signals the breakdown of the elastic behavior of the material while the limit surface is associated with the onset of unbounded flow.

For specific applications in which the material behavior needs to be accounted for with great accuracy the two surfaces noted above can be taken as a basis for a two-surface material model (5,6,11). In many cases, however, the simple one-surface model is often considered to be satisfactory and is preferred for its greater simplicity. In any case, an accurate representation of the surfaces involved is most desirable. In the next section, a number of experimental results are reviewed and a qualitative study is made of the features that subsequent yield and limit surfaces develop during a process of loading. From these results it will be shown that the usual isotropic-kinematic von Mises representations in some instances are quite unsatisfactory, particularly with regards to producing
accurate predictions of the direction of the plastic strain rates. A general formalism is then proposed that in a simple manner allows for distortional hardening in the material model, which appears to significantly improve the predicted direction of plastic flow under further loading.

A REVIEW OF EXPERIMENTAL RESULTS REGARDING YIELD AND LIMIT SURFACES

In this section a brief review of experimental results is presented that motivates the subsequently proposed material model. No attempt to completeness is made in this review. A thorough survey of experimental work in plasticity may be found in (17). The first set of results (10,13, 16,19,21,24) is concerned with the determination of yield surfaces based on the proportionality limit as definition of yield. Experimental work on limit surfaces, i.e., based on the back-extrapolation method, is very limited. The work by Mair and Pugh (14) is one of the few studies available in this subject.

![Graph showing yield surfaces for aluminum](image.png)

**Fig. 2.—Subsequent Yield Surfaces for an Aluminum Tubular Specimen. Reprinted with Permission of the American Society of Mechanical Engineers from "An Experimental Study on Initial and Subsequent Yield Surfaces in Plasticity," by P. M. Naghdi, F. Essenburg, and W. Koff, Journal of Applied Mechanics, Vol. 25, 1958, p. 206**
Naghdai, Essenburg and Koff (19) reported an experimental study consisting of two subsequent yield surfaces for aluminum tubular specimen. Their study covered the first and fourth quadrants of the axial-shear stress plane. The tubes were preloaded in pure torsion. The initial and two subsequent yield surfaces obtained are shown in Fig. 2. At the stress point, in the direction of positive shear stress, a region of high curvature reminiscent of a corner is observed to appear, while in the opposite direction a flattening of the yield surface takes place. Perpendicular to the direction of loading, the yield surface appears unaltered, i.e., no cross effect is observed. The same observations can be made regarding the results obtained by McComb (13), Fig. 3, which were also based on thin-walled aluminum tubes. The results by Ivey (10) and Phillips (21) show in addition a significant translation of the yield surface in the direction of loading, together with a moderate overall flattening in the same direction, Fig. 4. Shiratory et al. (24) reported experiments on thin-walled brass tubes subjected to axial tension, internal or external pressure and torsion. They determined subsequent yield surfaces for various loading directions. A significant cross-effect was observed in all cases, an example of which is shown in Fig. 5(a). Figure 5(b) illustrates the effect of the definition of yield on the outcome of the experiment. It is noted that the amount of offset plastic strain taken in the definition significantly affects the resulting yield surface which tends to shrink as the offset

![Graph showing initial and subsequent yield surfaces for an aluminum tubular specimen.](image-url)

**FIG. 3.—Initial and Subsequent Yield Surfaces for an Aluminum Tubular Specimen. Reprinted with Permission of the National Aeronautics and Space Administration from Some Experiments Concerning Subsequent Yield Surfaces by H. G. McComb, TND-396, 1960**

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plastic strain is decreased. This can be viewed as an important reason in favor of the back-extrapolation method which is independent of any definition of the onset of yield and is based on a more easily discernible property of the material.

The results by Mair and Pugh (14) illustrate the behavior of subsequent limit surfaces, defined by back-extrapolation. Their experiments made use of thin-walled copper tubes subjected to a combination of torsion and extension. The limit surface was found to undergo expansion, translation and curvature distortions of a nature similar to those observed in yield surfaces defined by means of the proportionality limit, although significantly milder, Fig. 6.

The experimental data presented above warrant the following general conclusions regarding the evolution of yield and limit surfaces:

1. The elastic region experiences a moderate isotropic expansion or does not expand at all.
2. The elastic region translates in stress space in the direction of the plastic strain rates.
3. A region of high curvature develops in the direction of loading.

4. Opposite to the direction of loading, a flattening of the surface takes place.

5. Rise of the yield stress in the directions perpendicular to the direction of loading (cross-effect) is moderate or non-existent.

Despite an overall similarity, it should be emphasized that the aforementioned curvature distortions are significantly higher in yield surfaces than in limit surfaces. The latter, however, experience a more pronounced overall expansion and cross-effect.

GENERAL DISTORTION HARDENING RULES FOR METAL PLASTICITY

Most plasticity models used in practice are based upon an isotropic-kinematic hardening assumption and therefore neglect effects 3, 4, and 5 (7,23,29). While the overall size and location of the elastic region can be satisfactorily matched by means of an isotropic-kinematic model, the direction of the plastic strain rates, being normal to the yield or limit surface, is greatly influenced by the observed changes in curvature.

In this section, general expressions for yield and limit surfaces are developed that include the isotropic-kinematic von Mises model as a particular case and that can be systematically used to incorporate distortional hardening features into the material modeling. The basis for the proposed formalism is a general Fourier expansion involving the angles between the stress tensor and certain "directors" defining the orientation of the various components of distortional hardening. Different models are then obtained by retaining selected terms in the expansion. For instance, retaining only the constant term results in the isotropic-kinematic von Mises model. The second harmonic in the expansion accounts for the cross-effect and the third one can be used to model flat and high curvature regions. This is an improvement over previously proposed
distortional hardening rules which were restricted to elliptical elastic regions \((2,3,26)\). More complicated hardening rules can conceivably be formulated by retaining additional terms.

These expressions are complemented with suitable rate equations for the parameters involved that are based on simple and well-established experimental facts. These rate equations and the parametric studies included herein are oriented towards modeling limit surfaces and therefore apply to one-surface models. The same techniques, however, can be used to model yield surfaces as well, which together with the results presented here can be used as a basis for more refined two-surface models.

Some useful notation is introduced first. Consider two second order symmetric tensors \(\alpha\) and \(\beta\), with components \(\alpha_{ij}, \beta_{ij}, i, j = 1, 2, 3\). The inner product of \(\alpha\) and \(\beta\) can be expressed as

\[
\langle \alpha, \beta \rangle = \frac{1}{2} \alpha_{ij} \beta_{ij} \tag{1}
\]

where the summation convention is implied. The norm associated with this inner product takes the form

\[
\|\alpha\| = \langle \alpha, \alpha \rangle^{1/2} \tag{2}
\]
In the derivations that follow the notion of angle between tensors is frequently used. This can be defined, with the aid of the inner product given by Eq. 1 as

$$\langle \alpha, \beta \rangle = \arccos \left( \frac{\langle \alpha, \beta \rangle}{\|\alpha\| \|\beta\|} \right)$$  \hspace{1cm} (3)

where angle $\langle \alpha, \beta \rangle$ denotes the angle between $\alpha$ and $\beta$.

With this notation, the isotropic-kinematic von Mises yield criterion can be expressed as

$$\|s - \alpha\| = k$$  \hspace{1cm} (4)

in which $s$ denotes the deviatoric part of the stress tensor, $\alpha$ is the deviatoric back-stress tensor which determines the position of the center of the elastic domain and $k$ is the yield shear stress. Expression 4 can be simplified by defining the "effective" deviatoric stress tensor $\tilde{s} = s - \alpha$ leading to

$$\|\tilde{s}\| = k$$  \hspace{1cm} (5)

The back-stress $\alpha$ is known to arise as a result of the heterogeneity of plastic deformation at a microscopic level (27,28). Both physical (12) and phenomenological considerations (23,29) lead to the following rate equation for the evolution of $\alpha$

$$\dot{\alpha} = H_0 \dot{\varepsilon}^p$$  \hspace{1cm} (6)

where $\dot{\varepsilon}^p$ denotes the plastic strain rates and $H_0$ is a plastic modulus. In the simplest case $H_0$ is constant, which leads to a bilinear stress-strain diagram. Curved bounding lines can be obtained, for instance, by allowing $H_0$ to be a function of $\|\alpha\|$.

As is well-known, the isotropic-kinematic model allows for expansion and translation of the yield surface but not for its curvature distortions. To implement the latter feature into the material model, a class of distortional hardening rules can be obtained by means of the following yield criterion

$$\|\tilde{s}\| = k \left( 1 + \sum_{n=2}^{\infty} \rho_n \cos n\theta_n \right)$$  \hspace{1cm} (7)

in which $\tilde{s}$ and $k$ retain the same meaning as in Eq. 5, $\rho_n$, $n = 2, 3, \ldots$ are scalar coefficients and $\theta_n$, $n = 2, 3, \ldots$ are the angles that the effective stress tensor $\tilde{s}$ makes with a sequence of unit tensors $\beta_n$, $\|\beta_n\| = 1$, $n = 2, 3, \ldots$, or "directors." In other words,

$$\theta_n = \arccos (\tilde{s}, \beta_n)$$  \hspace{1cm} (8)

The set of quantities $\rho_n$, $\beta_n$, $n = 2, 3, \ldots$ can be viewed as a set of phenomenological internal variables. The meaning of these variables can be illustrated by examining separately the effect that each term in the expansion of Eq. 7 exerts on the yield surface. For instance, by retaining only the first term, i.e., by setting

$$\|\tilde{s}\| = k(1 + \rho_2 \cos 2\theta_2)$$  \hspace{1cm} (9)

a yield surface is obtained like the one shown in Fig. 7. For a negative
value of $\rho_2$, the yield surface is seen to flatten in the direction of $\beta_2$ and widen in the perpendicular direction. Thus, the first term in the expansion of Eq. 7 introduces a cross-effect type of distortion in the yield surface in which the parameter $\rho_2$ determines the magnitude of the distortion and $\beta_2$ its orientation. This component of the distortion of the yield surface will be referred to as "binary distortion."

Retaining the second term in the expansion, i.e., setting

$$\|s\| = k(1 + \rho_3 \cos 3\theta_3)$$

(10)

a different type of distortion is obtained, Fig. 8, in which a region of higher curvature appears in the direction $\beta_3$, together with a flattening in the opposite direction and no cross-effect. This distortional component will be referred to as "ternary" distortion. As before, the amount of ternary distortion is determined by the parameter $\rho_3$ and its orientation by $\beta_3$.

Retaining more terms in the expansion, more complicated shapes can be modeled. For limit surfaces, however, in which the curvature distortions are mild, excellent results are achieved with binary and ternary distortion only, i.e., with a yield criterion

$$\|s\| = k(1 + \rho_2 \cos 2\theta_2 + \rho_3 \cos 3\theta_3)$$

(11)

In the remainder of this section, a parametric study is carried out that together with simple experimental observations suggests expressions and rate equations for $\rho_2$, $\rho_3$, $\beta_2$ and $\beta_3$.

The experimental observations noted earlier indicate that the translational, binary and ternary hardening of the limit surface all take place in the same direction, namely, along the direction of loading or of plastic deformation. This suggests setting
where $\hat{\alpha}$ denotes the unit tensor along $\alpha$. Under these assumptions, rate Eq. 6 determines the evolution of $\alpha$, $\beta_2$ and $\beta_3$. Therefore only the rate equations for $\rho_2$ and $\rho_3$ remain to be determined.

To this end, one first notes that the yield criterion, Eq. 7, determines a convex elastic region only for values of the parameters within a certain range. It is important, therefore, that the rate equations for $\rho_2$ and $\rho_3$ insure that their values always remain within this admissible range. Straightforward geometric considerations (see Appendix I) yield the following condition for convexity of the elastic domain defined by Eq. 11

$$5|\rho_2| + 10|\rho_3| \leq 1$$

Introducing the constitutive assumption that the magnitudes of binary and ternary distortion are proportional to each other simplifies the problem further. This defines

$$\rho_2 = r_2 \rho; \quad \rho_3 = r_3 \rho$$

where $\rho$ is a scalar variable and $r_2$ and $r_3$ are proportionality constants which can be normalized by the condition $\sqrt{r_2^2 + r_3^2} = 1$. With this additional assumption, the convexity condition, Eq. 13, becomes

$$|\rho| \leq \frac{1}{5|r_2| + 10|r_3|} \equiv \rho_\infty$$

where $\rho_\infty$ is a limiting value of $\rho$ that cannot be surpassed in order to preserve convexity.
A suitable constitutive assumption for $\rho$ that satisfies the convexity condition consists of taking $\rho$ to be a function of $\|\alpha\|$ which has a horizontal asymptote at $\rho_\infty$, i.e.,

$$\rho = f(\|\alpha\|) = f(\alpha) \quad \text{.................................................................................................................. (16)}$$

where the function $f$ is of the type shown in Fig. 9. For simplicity of notation, henceforth let $\alpha = \|\alpha\|$. Equation 16 can be expressed in rate form as follows

$$\dot{\rho} = f'(\alpha) \dot{\alpha} = f'(\alpha) H_0(\alpha) \langle \dot{\alpha}, \dot{\varepsilon}^p \rangle = h(\rho) \dot{\gamma}^p \quad \text{.................................................................................................................. (17)}$$

in which one writes

$$h(\rho) = f'[f^{-1}(\rho)] H_0[f^{-1}(\rho)] \quad \text{.................................................................................................................. (18)}$$

and

$$\dot{\gamma}^p = \langle \dot{\alpha}, \dot{\varepsilon}^p \rangle \quad \text{.................................................................................................................. (19)}$$

As an example, if one chooses the function $f$ to be of the exponential type

$$f(\alpha) = \rho_\infty (1 - e^{-\alpha/\alpha_c}) \quad \text{.................................................................................................................. (20)}$$

in which $\alpha_c$ is a characteristic value of $\alpha$, Fig. 9, then $h$ can be expressed as

$$h(\rho) = \frac{1}{\gamma_c} (\rho_\infty - \rho) \quad \text{.................................................................................................................. (21)}$$

in which one writes $\gamma_c = \frac{\alpha_c}{H_0}$, and the rate Eq. 17 reduces to

$$\dot{\rho} = \frac{\dot{\gamma}^p}{\gamma_c} (\rho_\infty - \rho) \quad \text{.................................................................................................................. (22)}$$

In this expression, the asymptotic character of the value $\rho_\infty$ becomes apparent. Note that, once the rate equations for the parameters $\alpha_c$, $\beta$, and $\rho_\infty$ have been specified, the evolution of $k$ automatically follows from the classical consistency condition which states that the stress trajectory must remain on the yield surface during a process of plastic loading.

It is interesting to note that the above equations can be readily ex-

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**FIG. 9.—Plausible Dependence of Distortion Magnitude on Back-Stress $\alpha$**

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tended to more complicated models in which more terms in the expansion of Eq. 7 are retained. In any case, the only independent variables remaining are $\alpha$ and $k$ so that the computational effort involved in using the model is comparable to that of the isotropic-kinematic model. In terms of parameter identification, the only additional parameters are $r_1$, $r_2$ and $a_c$ or $\gamma_c$. In the numerical examples that follow typical values of these parameters are given and the ability of the model to describe the distortional hardening of limit surfaces is demonstrated.

Figure 10 shows some analytical curves superimposed on the experimental data by Mair and Pugh (14). All results refer to the $(\sigma, \tau)$ plane, where $\sigma$ and $\tau$ denote the axial and shear stress, respectively. In this simple case, the norm in Eq. 2 takes the form

$$
\begin{bmatrix}
\sigma \\
\tau
\end{bmatrix} = \sqrt{\sigma^2 + 3\tau^2}
$$

(23)

The proportionality constants used in Fig. 10 were $r_2 = -0.949$ and $r_3 =$

FIG. 10.—Comparison of Analytically Generated Yield Surfaces with Experimental Data of Mair and Pugh for Copper Tubular Specimen
0.316 which substituted into Eq. 15 result in $\rho_\infty = 0.126$. The innermost yield curve corresponds to normalized values $k = 1$, $\alpha = (0, 0)$ and $\rho = 0$, which define the usual von Mises yield surface. The two subsequent yield curves correspond to $k = 1.440$, $\alpha = (0, 0.092)$, $\rho = 0.063$ and $k = 1.880$, $\alpha = (0, 0.184)$, $\rho = 0.126$, respectively. These values correspond to taking $\alpha_c = 0.319$ and the bilinear envelope in Fig. 9 as the $\rho$-$\alpha$ curve, and to adopting a bilinear shear stress-shear strain ($\tau$-$\gamma$) diagram with plastic modulus $H_p$ and then taking $H_p = 1.053H_p'$ in Eq. 6. Using these parameters, the analytic curves shown in Fig. 10 are the result of subjecting the model to a proportional stress path in the $\tau$-axis direction, from $\tau = 0$ to $\tau = 1.183$. Figure 10 shows an excellent agreement between experimental and analytical results. Given the scarcity of experimental data based on the back-extrapolation method it is difficult to assess the accuracy of the proposed model for other loading conditions. It is hoped that future experimental work will fill in this gap and make possible a more thorough assessment of the model.

**Summary and Conclusions**

A phenomenological model for inelastic behavior of metals suitable for random cyclic loading is described in this paper. General expressions for yield and limit surfaces are developed that include the isotropic-kinematic von Mises model as a particular case and that can be systematically used to incorporate distortional hardening features into the material modeling. The basis for the proposed formalism is a Fourier-type expansion involving the angles between the stress tensor and certain directors defining the orientation of the various components of distortional hardening. For limit surfaces, defined by means of the back-extrapolation method and which exhibit mild curvature distortions, a simplified two term expansion yields results which are in excellent agreement with the limited available experimental data. The only independent plastic variables involved in this simplified model are the translation and radius of the elastic domain. This makes the computer implementation of the proposed model no more difficult than that of the familiar isotropic-kinematic model, yet gaining much in accuracy.

Two kinds of refinements can be easily introduced to make the model more accurate. The number of terms in the expression for the yield criterion can be increased permitting the description of severely distorted surfaces, such as subsequent yield surfaces defined by means of the proportionality limit. From the available data, such a refinement appears to be unnecessary for limit surfaces. Secondly, the formalism proposed herein can be used for developing a more refined two surface model with simultaneous distortional hardening of both the yield and the limit surface. It is believed, however, that in the majority of practical applications the basic model suggested in this paper should suffice.

**Acknowledgments**

This paper is based on research conducted under Grant No. CEE-81-07217 from the National Science Foundation.
In this appendix, a brief derivation of inequality given by Eq. 13 is presented. Equation 13 expresses a unilateral constraint acting on parameters \( \rho_2 \) and \( \rho_3 \), whose satisfaction ensures convexity.

The limit surface under consideration is given by Eq. 11, and can be viewed as a 5-dimensional hypersurface in 6-dimensional stress space, say \( \Sigma \). By construction, however, this hypersurface is a cylinder oriented along the hydrostatic axis. Therefore, attention can be confined to a 4-dimensional deviatoric section of \( \Sigma \), say \( \Sigma' \). Furthermore, since according to Eq. 12 \( \beta_2 = \beta_3 = \hat{\alpha} \), an inspection of Eqs. 8 and 11 reveals that \( \Sigma' \) is axisymmetric about the axis \( \hat{\alpha} \). Thus, for the purpose of assessing convexity, it suffices to consider a 2-dimensional meridional section of \( \Sigma' \).

In polar coordinates this is given by

\[
    r(\theta) = 1 + \rho_2 \cos 2\theta + \rho_3 \cos 3\theta \quad \cdots \quad (24)
\]

The curvature \( \kappa \) of this curve at point \( \theta \) is given by the expression

\[
    \kappa(\theta) = \frac{-rr'' + 2r'^2 + r^2}{(r'^2 + r^2)^{3/2}} \quad \cdots \quad (25)
\]

in which \( r'(\theta) = \frac{dr(\theta)}{d\theta} = -2\rho_2 \sin 2\theta - 3\rho_3 \sin 3\theta \)

\[
    r''(\theta) = \frac{d^2r(\theta)}{d\theta^2} = -4\rho_2 \cos 2\theta - 9\rho_3 \cos 3\theta \quad \cdots \quad (26)
\]

For \( \rho_2 = \rho_3 = 0 \), the meridional curve given by Eq. 24 reduces to the unit circle, with curvature identically equal to 1. Convexity will be maintained for nonzero \( \rho_2 \) and \( \rho_3 \) as long as the curvature \( \kappa \) is everywhere greater or equal to zero, i.e.,

\[
    \kappa(\theta) \geq 0 \quad \text{for every} \quad \theta \in [0, 2\pi] \quad \cdots \quad (27)
\]

The choice \( \rho_2 \leq 0 \) and \( \rho_3 \geq 0 \) suggested by experimental evidence (also see discussion following Eqs. 9 and 10) implies that \( \kappa \) has a minimum at \( \theta = \pi \), as inspection of Figs. 7 and 8 reveals. But at this point

\[
    r(\pi) = 1 + \rho_2 - \rho_3 \quad r'(\pi) = 0 \quad r''(\pi) = -4\rho_2 + 9\rho_3 \quad \cdots \quad (28)
\]

and the convexity condition given by Eq. 27 with the aid of Eq. 25, becomes

\[
    r(\pi) \geq r''(\pi) \quad \cdots \quad (29)
\]

Substituting Eq. 28 into Eq. 29 one obtains

\[
    -5\rho_2 + 10\rho_3 \leq 1 \quad \cdots \quad (30)
\]

Finally, recalling the assumption \( \rho_2 \leq 0 \) and \( \rho_3 \geq 0 \), it follows that \( -\rho_2 = |\rho_2| \) and \( \rho_3 = |\rho_3| \), so that Eq. 30 may be expressed as

\[
    5|\rho_2| + 10|\rho_3| \leq 1 \quad \cdots \quad (31)
\]

which is the sought result, Eq. 13.

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**APPENDIX III.—NOTATION**

The following symbols are used in this paper:

- \(H_o\) = plastic modulus;
- \(h\) = distortional modulus;
- \(k\) = shear yield stress;
- \(r, \theta\) = polar coordinates of meridional section of limit surface;
- \(r_n\) = proportionality constants for hardening parameters;
- \(s = \) deviatoric stress tensor;
- \(\varepsilon = \) effective deviatoric stress tensor;
- \(\alpha = \) back-stress tensor;
- \(\alpha = \) back-stress norm;
- \(\hat{\alpha} = \) back-stress direction;
- \(\alpha_c = \) characteristic value of back-stress norm;
- \(\beta_n = \) hardening directors;
- \(\dot{\gamma} = \) effective plastic strain rate;
- \(\gamma_c = \) characteristic value of effective plastic strain;
- \(\dot{\varepsilon} = \) plastic strain rate;
- \(\kappa = \) curvature of meridional section of limit surface;
- \(\rho_n = \) hardening parameters;
- \(\rho = \) distortion magnitude;
- \(\rho_o = \) convexity limit for distortion magnitude;
- \(\sigma = \) axial stress;
- \(\tau = \) shear stress; and
- \(\theta = \) hardening orientation angles.