A statistical theory of polycrystalline plasticity

BY M. ORTIZ AND E. P. POPOV

Division of Structural Engineering and Structural Mechanics,
Department of Civil Engineering, University of California,
Berkeley, California 94720, U.S.A.

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The plasticity and viscoplasticity of polycrystalline materials are studied analytically in terms of lattice dislocations, with the principal effects attributed to non-extended obstacles. Non-equilibrium statistical mechanics is used to describe the evolution of the dislocation structures during loading and unloading processes. A plausible variation in the probability density function for mobile dislocations for such processes is suggested. The proposed material model is in good qualitative agreement with several observed phenomena that previously could not be quantified on the basis of the dislocation theory. Numerical examples illustrate the effect of the rate of loading, the variations in the recovery effect as it relates to the extent of load reversal, and a means for treating materials that exhibit a yield plateau. In the limit, the proposed model yields results for inviscid plasticity.

1. Formal framework

Fifty years of experimental observations have conclusively shown that plasticity and viscoplasticity are typical properties of crystalline materials, and that plastic deformation is the overall effect of structural changes that take place at the microscopic level during a process of loading. In particular, the propagation of dislocations seems to play a dominant role in this respect, and is considered in this paper.

It is well established that dislocations move inside well defined crystallographic planes, or glide planes. The set of all dislocations in parallel glide planes and with parallel Burgers vectors is referred to as a glide system. Thus, the orientation in space of a generic glide system can be defined by two orthogonal unit vectors, say \( \mathbf{n} \) and \( \mathbf{m} \), which represent the normal to the glide plane and the direction of the Burgers vector, respectively. In a single crystal, the glide systems form a finite set, closely related to the characteristics of the crystallographic lattice. However, in a polycrystalline material, glide plane orientations and Burgers vector directions are continuously distributed variables.

It is well established (Sackett et al. 1977) that, under these circumstances, the
macroscopic plastic deformation rate, say \( \dot{\varepsilon}_{ij}^p \), is related to the dislocation motion as follows:

\[
\dot{\varepsilon}_{ij}^p = \int \dot{\gamma}^p(n, m) \mu_{ij} \, d\mu,
\]

where

\[
\mu_{ij} = \frac{1}{2}(m_i n_j + m_j n_i)
\]

is a symmetric orientation tensor for the glide system, \( \dot{\gamma}^p(n, m) \) is the plastic shear strain rate due to the motion of the dislocations in the glide system \((n, m)\), and the integral extends to all glide system orientations. The plastic shear strain rate \( \dot{\gamma}^p \) of a generic glide system (Kröner & Teodosiu 1972) is given by

\[
\dot{\gamma}^p = b \rho \bar{v},
\]

where \( b \) is the length of the Burgers vector, \( \rho \) is the mobile dislocation density of the glide system and \( \bar{v} \) is the average velocity of expansion of the dislocations, given by

\[
\bar{v} = \frac{1}{l} \int_v dl,
\]
i.e. \( \bar{v} \) is the average velocity over the total length \( l \) of mobile dislocation lines in a glide system with respective expansion velocities \( v \) for each of the dislocation segments \( dl \).

The mobile dislocation density \( \rho \) is known to change its value during a process of loading. This is caused by breeding due to multiple cross glides and regeneration due to the Frank–Read sources (Frank & Read 1950) which take place simultaneously with pair annihilation. Reasonably satisfactory kinetic equations have been proposed (Gilman 1969) that describe the evolution of \( \rho \). Studies made on this basis indicate that under increasing load a saturation stage is reached when the multiplication and annihilation mechanisms compensate each other, resulting in a constant value of the dislocation density. In this treatment, for simplicity, the dislocation density will be considered constant.

By forming the plastic dissipation expression with the aid of (1), one has

\[
\dot{W}^p = \sigma_{ij} \dot{\varepsilon}_{ij}^p = \int \dot{\gamma}^p(n, m) \sigma_{ij} \mu_{ij} \, d\mu.
\]

It follows from (5) that the variable conjugate to \( \dot{\gamma}^p \) is

\[
\tau = \sigma_{ij} \mu_{ij}.
\]

By using (2), it can be shown readily that \( \tau \) is the resolved shear stress acting on the glide plane in the direction of the Burgers vector.

In the sequel, it will be assumed that the response of a given glide system, given by \( \dot{\gamma}^p \), is dependent only on \( \tau \) and the state of hardening of that particular system. Interaction between the glide systems is ignored, and temperature effects are not

† The present formulation can be extended to the case of finite deformations by making use of the concept of the multiplicative decomposition of the deformation gradients (Lee & Liu 1967, Mandel 1972).
considered in this discussion. Thereby, the problem is reduced to that of the study of a single glide system and the integral effect of such systems in the sense of expression (1).

The motion of dislocations in pure slip in any one glide system appears to have a viscous character. For example, Lothe (1962) estimated analytically the mobility during glide of uniformly moving dislocation segments and demonstrated the existence of a relation between \( v \) and \( \tau \) of the type

\[
v = \frac{\tau}{\eta},
\]

where \( \eta \) is a temperature-dependent viscosity coefficient. For a dislocation segment moving through a distribution of obstacles, the quantity \( \tau \) has to be interpreted as the effective shear stress acting on the dislocations, i.e. the applied resolved shear stress less the resistance posed by the obstacles.

Dislocations and obstacles, which cause hardening, are found in crystalline materials in such large numbers that, although the equations of motion of an elementary dislocation segment can be expressed in deterministic terms, the study of a dislocation ensemble itself can be treated only by using a probabilistic approach. Several authors have based their developments on this fact (Sackett et al. 1977, Kocks 1966, Kröner 1970), but a complete theory is not as yet available. This paper attempts to provide some advance in this direction.

2. Interaction between dislocations and obstacles

If the macroscopic plastic deformation of polycrystalline materials is the result of the overall motion of dislocations, the loading-unloading irreversibility, which characterizes plastic materials, is the result of the peculiar way in which dislocations interact with the obstacles present in the glide planes. These obstacles are very varied. In this paper only non-extended obstacles whose interactions with dislocations are effective only over a few atomic distances are considered. To this group belong impurity atoms, precipitates, jogs in the glide dislocations, and forest dislocations.

Non-extended obstacles often have been idealized as point obstacles. Dislocation segments arrested by a pair of such adjoining point obstacles, defining a line or a link between them, require a certain shearing stress to surmount these obstacles (figure 1). The strength or the resistance offered by a generic link to the flow of dislocations past it will be designated here by \( s \). In general, \( s \) depends on the length of the link and the physical properties of the obstacles (Foreman & Makin 1966, 1967). Since the distribution of point obstacles on a glide plane is entirely random (Mughrabi 1975, Grosskreutz & Mughrabi 1975), it follows that \( s \) itself is also a random variable. Herein \( f(s) \) will denote the probability density function for the strength of the links in a generic glide plane. For simplicity, this probability density function is assumed to be constant and independent of the state of hardening of the glide system. As will become apparent later, different forms of \( f(s) \) result in different
shapes of stress–strain curves. Therefore, \( \tilde{f}(s) \) can be thought of as a property of the material.

Consider now a generic glide system with a dislocation density \( \rho \). At any given time \( t \), the dislocation population consists of segments that are distributed over links of different strengths (figure 1). At such a time, \( f(s, t) \) will be used to designate the fraction of the dislocation density facing links of strength \( s \). This probability density function completely defines the state of hardening of the glide system. For instance, with the aid of (3), the plastic strain rate can be expressed as

\[
\dot{\gamma}^p(t) = \int_0^\infty b \rho \left\langle \frac{\tau(t) - s}{\eta} \right\rangle f(s, t) \, ds,
\]

where \( \left\langle \tau(t) - s \right\rangle \) is the effective shearing stress acting on dislocation segments facing links of strength \( s \). The angle brackets indicate that, if the resistance of the obstacles is greater than the stress applied on a dislocation segment, any further movement of the segment cannot take place. According to (7), the quantity

\[
v(s, t) = \left\langle \tau(t) - s \right\rangle / \eta
\]

is the instantaneous velocity of dislocation segments facing links of strength \( s \). Comparing (3) with (8), one can note that the average expansion velocity of the dislocations of the glide system is taken as

\[
\bar{v}(t) = \int_0^\infty v(s, t) f(s, t) \, ds,
\]

i.e. the average given by (4) is replaced by the ensemble average.

The probability density functions of \( \tilde{f}(s) \) and \( f(s, t) \) are dimensionally alike, but they represent two completely different concepts. While \( \tilde{f}(s) \) characterizes the strength of the obstacles that can be found in a generic glide plane, the function \( f(s, t) \) describes how the dislocations are distributed relative to the obstacles at a given time. Nevertheless, they may occasionally be equal. For instance, in a virgin material, dislocations can be assumed to have been driven by thermal activation to random locations over the obstacles, so that \( f(s, t) \) can be initially taken to equal \( \tilde{f}(s) \) (an exception to this is furnished by a material exhibiting an initial plateau in its stress–strain diagram, see §6).
The function \( f(s, t) \) will vary substantially during a process of loading. For instance, if the stress level is suddenly raised from 0 to \( \tau \) and kept constant thereafter, all the dislocations originally facing links of strengths less than \( \tau \) will move until they eventually become arrested at links of strength \( s \geq \tau \). Kinetic equations that govern the time evolution of the function \( f(s, t) \) are proposed in the next section.

It should be noted carefully that the density function \( f(s, t) \) representing the current distribution giving strengths for links faced by the dislocation segments when moving in one direction, such as \( m \) in figure 2, will in general be different from the one giving strengths for the links opposing the motion of the dislocations in the opposite direction, such as \(-m\). Therefore, it is essential to consider the glide systems \((n, m)\) and \((n, -m)\) as two different ones, in spite of the fact that the dislocation segments in both of them are physically the same.

\[ \text{Figure 2. Dislocation line movement through point obstacles in two opposite directions, } +m, -m. \]

3. Kinetic equations

Consider a generic glide system with a resolved shear stress \( \tau(t) \) and denote by \( \bar{l} \) the average spacing between point obstacles. Then the time it takes for a dislocation segment facing a link of strength \( s \) to jump to another link is

\[ \Delta t = \bar{l}/v(s, t), \quad (11) \]

where \( v(s, t) \) is given by (9). The probability that the new link has strength \( s' \) is given precisely by \( f(s') \). Hence, the transition probability rate at time \( t \) that a dislocation segment makes a jump from a link of strength \( s \) to one of strength \( s' \) is

\[ \dot{\psi}(s \rightarrow s'; t) = v(s, t)f(s')/\bar{l} = \langle \tau(t) - s \rangle f(s')/\bar{l}. \quad (12) \]

Therefore, on the assumption that the process is Markovian, the evolution for \( f(s, t) \) is given by Pauli’s master equation (Blanc-Lapierre & Fortet 1957)

\[ \frac{\partial f(s, t)}{\partial t} = \int_0^\infty [\dot{\psi}(s' \rightarrow s; t)f(s', t) - \dot{\psi}(s \rightarrow s'; t)f(s, t)] ds'. \quad (13) \]
Substituting (12) into (13), one obtains

\[
\frac{\partial f(s, t)}{\partial t} = \frac{\tilde{f}(s)}{\eta \bar{l}} \int_0^\infty \langle \tau(t) - s' \rangle f(s', t) \, ds' - \frac{\langle \tau(t) - s \rangle}{\eta \bar{l}} f(s, t). \tag{14}
\]

At this point, it is convenient to normalize stresses with respect to some characteristic quantity \( \bar{s} \), for instance the average value of \( s \) with respect to \( \tilde{f}(s) \). For this purpose, introduce the following notation:

\[
r(t) = \frac{\tau(t)}{\bar{s}}, \quad x = \frac{s}{\bar{s}}. \tag{15}
\]

With this notation, (14) becomes

\[
\frac{\partial f(x, t)}{\partial t} = \frac{\tilde{f}(x)}{t_c} \int_0^\infty \langle r(t) - x' \rangle f(x', t) \, dx' - \frac{\langle r(t) - x \rangle}{t_c} f(x, t), \tag{16}
\]

where

\[
t_c = \eta \bar{l} / \bar{s} \tag{17}
\]

is a characteristic time for the process. For completeness, (8) can now be rewritten in normalized quantities as

\[
\dot{\gamma}^p(t) = \frac{\gamma_0^p}{t_c} \int_0^\infty \langle r(t) - x \rangle f(x, t) \, dx, \tag{18}
\]

where

\[
\gamma_0^p = b \rho \bar{l} \tag{19}
\]

is a characteristic plastic shear strain.

Kinetic equation (16) can be interpreted as follows: the second term removes dislocation density from the interval \( 0 < x < r(t) \), and the first term distributes it over the interval \([0, \infty]\), proportionally to \( \tilde{f}(x) \). Part of the removed dislocation density goes to stable positions (interval \([r(t), \infty]\)) and part of it is fed back into the segment \([0, r(t)]\) again, being then recycled (figure 3).
To illustrate this type of behaviour, the evolution of \( f \) is shown in figure 4, for a loading history \( r(t) = rH(t) \), i.e. for a sudden jump in the applied stress from 0 to \( r \) at \( t = 0 \) with \( r(t) \) being kept constant thereafter. For this case, it follows from the previous discussion that the function \( f \) approaches asymptotically the limit:

\[
f_0(x, r) = \lim_{t \to \infty} f(x, t) = \left[ f(x, t_0) + \frac{\tilde{f}(x) P(r, t_0)}{1 - \tilde{P}(r)} \right] H(x - r),
\]

(20)

\[\text{Figure 4. Example of transient response from an initial distribution } f(x, 0) \text{ a } \beta \text{-function, }\]

\[p = 2, q = 3 \text{ to an equilibrium distribution as } t/t_0 \to \infty.\]

where \( P(x, t) \) and \( \tilde{P}(x) \) are the distribution functions corresponding to \( f(x, t) \) and \( \tilde{f}(x) \) respectively; \( f(x, t_0), P(x, t_0) \) are the initial values of \( f(x, t) \) and \( P(x, t) \), respectively, at time \( t = t_0 \); and \( H \) is the Heaviside step function.

One can note that \( f_0(x, r) \) in (20) is an equilibrium solution of the kinetic equation (16), i.e. one that renders \( \partial f / \partial t \equiv 0 \). Further, from (18) it follows that, when the equilibrium state \( f_0(x, r) \) is reached, any arbitrary loading process in the interval \([0, r]\) will yield no further plastic deformation. Thus, the interval \([0, r]\) behaves as an induced elastic region.
Consider next a monotonic quasistatic loading process $r(t)$ for which the characteristic time of loading is much larger than $t_c$. Then

$$f(x, t) = f_{0}[x, r(t)] = f_{0}(x, t).$$  \hspace{1cm} (21)

One can further note that $f_{0}(x, t)$ satisfies the following equation:

$$\frac{\partial f_{0}(x, t)}{\partial t} = \left(\tilde{f}(x) \frac{H[x - r(t)]}{1 - \tilde{P}[r(t)\hat{r}(t)]} - \delta(x - r(t))\right) f_{0}(r(t), t) \hat{r}(t).$$ \hspace{1cm} (22)

It also follows from the previous discussion that unloading is elastic. Thus, for an arbitrary (non-monotonic) loading process, (22) becomes

$$\frac{\partial f_{0}(x, t)}{\partial t} = \left(\tilde{f}(x) \frac{H[x - r(t)]}{1 - \tilde{P}[r(t)\hat{r}(t)]} - \delta(x - r(t))\right) f_{0}(r(t), t) \langle \hat{r}(t) \rangle$$ \hspace{1cm} (23)

where $\delta$ is a Dirac delta function. Equation (23) is rate independent and can be considered as the inviscid limit of kinetic equation (16) as $t_c \to 0$.

Some insight can be gained about the nature of this equation if one rewrites it by means of the same statistical arguments that were used to derive (16). Assume that, at time $t$, the normalized shear stress is $r(t)$, and the equilibrium density function is $f_{0}(x, t)$. Then, for $x \leq r(t), f_{0}(x, t) = 0$. Consider now a time increment $dt$ over which the applied stress is incremented by $\hat{r}(t) dt$. If $\hat{r}(t) \leq 0, f_{0}$ remains unchanged. However, if $\hat{r}(t) > 0$, the dislocations in the interval $[r(t), r(t) + \hat{r}(t) dt]$ become unstable and move to stable links with strength $x > r(t)$. Therefore, the transition probability that a dislocation segment jumps from a link of strength $x$ to another one of strength $x'$ is given by:

$$\frac{\tilde{f}(x') H[x - r(t)]}{1 - \tilde{P}[r(t)\hat{r}(t)]} \delta(x - r(t)) \langle \hat{r}(t) \rangle dt,$$ \hspace{1cm} (24)

and the corresponding transition probability rate becomes

$$\psi_{0}(x \to x'; t) = \frac{\tilde{f}(x') H[x' - r(t)]}{1 - \tilde{P}[r(t)\hat{r}(t)]} \delta(x - r(t)) \langle \hat{r}(t) \rangle.$$ \hspace{1cm} (25)

By substituting (25) into the master equation (13), it can be shown that kinetic equation (23) can be obtained.

Equation (23) can be again interpreted as follows. The second term removes the dislocation density from the interval $[r(t), r(t) + \hat{r}(t) dt]$, and the first term distributes it over the stable interval $[r(t) + \hat{r}(t) dt, \infty]$, proportionally to $\tilde{f}(x)$ (figure 5).

It is of interest to obtain the inviscid limit of (18), giving the plastic strain rate. Once again, this can be established on physical grounds. The dislocation density in the interval $[r(t), r(t) + \hat{r}(t) dt]$ released by a stress increment $\hat{r}(t) dt$ gives rise to a plastic strain which can be written as

$$d\gamma^p = b \rho f_{0}[r(t), t] \hat{r}(t) dt \bar{n} \hat{t},$$ \hspace{1cm} (26)
where as before $\bar{l}$ is the average spacing between point obstacles and $\bar{n}$ is the average number of jumps that the dislocation segments make before reaching stable positions. The value of $\bar{n}$ can be determined as follows. The probability that a segment gets arrested at the first jump is given by $1 - \bar{P}[r(t)]$, and the probability that it goes beyond is $\bar{P}[r(t)]$. Then the probability that a segment gets arrested at the second jump is $\bar{P}[r(t)]\{1 - \bar{P}[r(t)]\}$, and the probability that it goes farther is $\bar{P}^{2}[r(t)]$, and so on. Therefore, the average number of jumps taken by the moving segments is

$$\bar{n} = \{1 - \bar{P}[r(t)]\} + 2\bar{P}[r(t)]\{1 - \bar{P}[r(t)]\} + 3\bar{P}^{2}[r(t)]\{1 - \bar{P}[r(t)]\} + \ldots .$$

(27)

This is a series whose sum is

$$\bar{n} = 1/(1 - \bar{P}[r(t)]).$$

(28)

Hence, substituting (28) into (26), with the aid of (19) one obtains the following expression for the plastic shear strain rate:

$$\dot{\gamma}^p = \frac{\gamma_{0}^p}{1 - \bar{P}[r(t)]}f_{0}[r(t), t] \dot{r}(t).$$

(29)

This equation is again rate-independent, and, if $\dot{f}(x)$ is non-zero over a finite interval $[0, x_{\text{max}}]$ only, then it predicts unbounded flow at $r = x_{\text{max}}$, since $\bar{P}(x_{\text{max}}) = 1$.

It should be remarked that the inviscid limits for (16) and (18), occurring when $\epsilon \rightarrow 0$, leading to (23) and (29) were found on physical grounds. Although this provides some insight into the physical processes involved, a rigorous mathematical proof would be more satisfactory and will be discussed in a subsequent paper.
4. Recovery

Consider a given glide system \((\mathbf{n}, \mathbf{m})\) and the corresponding density function \(f(x, t)\), giving the distribution of strengths of the links faced by the dislocations when moving in the direction \(\mathbf{m}\). As was remarked before, the dislocations in the glide systems \((\mathbf{n}, \mathbf{m})\) and \((\mathbf{n}, -\mathbf{m})\) are physically the same, but the corresponding \(f\)-functions are in general different. It is instructive to study how \(f(x, t)\) evolves

![Probability density curves](image)

**Figure 6.** Example of recovery from a perturbed initial distribution to \(f(x)\) a \(\beta\)-function, \(p = 2, q = 3\).

when the applied stress changes sign and the dislocations move in the opposite direction, \(-\mathbf{m}\). Here, it will be assumed that the unloading process begins at time \(t_0\) when \(f = f(x, t_0)\). Then as the dislocations move in the \(-\mathbf{m}\) direction, they gradually abandon the positions they held at \(t = t_0\), and the density function \(f(x, t)\) changes from its initial value \(f(x, t_0)\).

If \(\bar{v}(t)\) denotes the average velocity of the dislocations at time \(t\) in the \(-\mathbf{m}\) direction, the transition probability rate that a segment in the glide system \((\mathbf{n}, \mathbf{m})\) is removed from a link of strength \(x\) and taken to another one of strength \(x'\) is

\[
\psi(x \to x', t) = \frac{\bar{v}(t)}{\bar{l}} f(x').
\] (30)
Multiplying and dividing (30) by \( bp \), one obtains

\[
\dot{\psi}(x \to x', t) = \left[ \dot{\gamma}^p(t)/\gamma_c^0 \right] \tilde{f}(x'), \tag{31}
\]

where \( \dot{\gamma}^p(t) \) is the plastic shearing strain rate due to the motion of dislocations in the direction \( -m \) and \( \gamma_c^0 \) is a characteristic plastic shearing strain of the process. On substituting (31) into the master equation (13), and carrying out the integration, one finds

\[
\partial f(x, t)/\partial t = \left[ \dot{\gamma}^p(t)/\gamma_c^0 \right] [\tilde{f}(x) - f(x, t)]. \tag{32}
\]

Using the above reasoning, one notes from kinetic equation (32) that plastic deformation in the direction \( -m \) triggers a recovery of the density function \( f(x, t) \) that approaches the virgin distribution \( \tilde{f}(x) \) at an exponential rate (see §5 for the explicit expression). An illustration of this behaviour is shown in figure 6. Since equation (32) is rate independent, it applies equally well to both the viscoelastic and inviscid materials.

It is to be noted that plastic materials exhibit fading memory. An unloading process in a glide system gradually wipes out the effect of previous loading processes (represented by \( f(x, t_0) \) in (32)). The experimental evidence regarding the phenomenon of recovery is extensive and conclusive, in spite of which this effect does not seem to have been identified in the literature in the above manner before. Some illustrative examples and experimental results are discussed in §6.

### 5. Solutions of kinetic equations

The character and mathematical solutions of the kinetic equations introduced thus far are briefly discussed in this section.

The solution of (32), being a linear first-order differential equation, is direct and can be given as

\[
f(x, t) = f(x, t_0) \exp \left\{ -\left[ \gamma^p(t) - \gamma^p(t_0) \right]/\gamma_c^0 \right\} + \tilde{f}(x) \left[ 1 - \exp \left\{ -\left[ \gamma^p(t) - \gamma^p(t_0) \right]/\gamma_c^0 \right\} \right], \tag{33}
\]

where \( f(x, t_0) \) denotes the initial value of \( f(x, t) \) at \( t = t_0 \).

The solution of (22) has already been given by (20) and (21). Therefore, only the solution of kinetic equation (16) remains to be discussed. This can be done as follows. Define

\[
u(t) \equiv \int_0^\infty \langle r(t) - x \rangle f(x, t) \, dx \tag{34}
\]

and note that \( \dot{\gamma}^p = \gamma_c^0 u(t)/t_c \), whereupon (16) becomes

\[
\frac{\partial f(x, t)}{\partial t} = \frac{u(t)}{t_c} f(x) - \frac{1}{t_c} \langle r(t) - x \rangle f(x, t) \tag{35}
\]
which can be readily solved, yielding

\[
\begin{align*}
  f(x, t) &= f(x, t_0) \exp \left\{ - \left[ \int_{t_0}^{t} \langle r(t') - x \rangle dt' \right] / t_c \right\} \\
  &\quad + \left[ f(x / t_c) \right] \int_{t_0}^{t} u(t') \exp \left\{ - \left[ \int_{t'}^{t} \langle r(t'') - x \rangle dt'' \right] / t_c \right\} dt'.
\end{align*}
\] (36)
To evaluate this equation, one needs an expression for \( u(t) \). It can be obtained by substituting (36) into the kinetic equation (16), yielding

\[
\begin{align*}
  u(t) &= \bar{h}(t) + \frac{1}{t_c} \int_{t_0}^{t} k(t, t') u(t') \, dt',
\end{align*}
\]

where

\[
\begin{align*}
  \bar{h}(t) &= \int_{0}^{\infty} \langle r(t) - x \rangle \hat{f}(x) \exp \left\{ - \left[ \int_{t_0}^{t} \langle r(t') - x \rangle \, dt' \right] / t_c \right\} \, dx,
\end{align*}
\]

Figure 9. Uniaxial stress–plastic strain curve referred to the trace of internal stresses.

and

\[
\begin{align*}
  k(t, t') &= \int_{0}^{\infty} \langle r(t) - x \rangle \hat{f}(x) \exp \left\{ - \left[ \int_{t'}^{t} \langle r(t'') - x \rangle \, dt'' \right] / t_c \right\} \, dx.
\end{align*}
\]

Equation (37) is an inhomogeneous Volterra integral equation of the second kind of the type encountered in feedback systems. Its solution takes the form (Stakgold 1968)

\[
\begin{align*}
  u(t) &= \bar{h}(t) + \frac{1}{t_c} \int_{t_0}^{t} \gamma(t, t') \bar{h}(t') \, dt',
\end{align*}
\]

where \( \gamma(t, t') \) is the resolvent kernel, given by a Neumann series

\[
\begin{align*}
  \gamma(t, t') &= \sum_{n=1}^{\infty} k_n(t, t'),
\end{align*}
\]

where \( k_n(t, t') \) are the iterated kernels.

Equation (40) shows the heavily nonlinear functional dependence of the response of a glide system, \( \gamma^P(t) = \gamma^P_0 u(t) / t_c \), on the loading history \( r(t') \), \( t_0 \leq t' \leq t \). Hence, the material behaviour described by the kinetic equation (10) is that of a nonlinear material with memory.
6. Comparisons with Experimental Results

The function $f(x)$ in (16), (22), and (32) can be thought of as a material property, although attempts to determine it theoretically have been made. An example of the latter approach is illustrated in figure 7 (Kocks 1966). It would appear, however, that, because of the complexities involved, it is more realistic to study this problem by a system identification approach. In applying this procedure, theoretical considerations can provide valuable information on the basic features of $f(x)$, thereby considerably simplifying the identification process. Some examples for uniaxial stress states follow.

For example, an examination of numerous quasistatic uniaxial stress–strain curves for metals shows that such curves approach asymptotically a bounding line or a curve such as AA' in figure 8 (Dafalias 1975). Translating the curve AA' to the origin establishes the curve BB', from which a new effective stress $\sigma_{\text{eff}}$ may be measured. The resulting stress–strain curve for this stress, exhibiting an unbounded flow at the stress $\sigma_{\text{o}}$, is shown in figure 9. Moreover, a (strong) Bauschinger
effect appears upon loading as soon as the stress–strain curve crosses BB'. In terms of $\sigma^\text{eff}$, however, the Bauschinger effect appears as stress changes sign. The physical rationale involved in this operation can be explained if one assumes that an effective stress, $\tau$, acting on a glide system consists of two parts, i.e.

$$
\tau = \tau^\text{ext} + \tau^\text{int},
$$

(42)

![Figure 11. Example of normalized uniaxial stress–plastic strain curves in the inviscid limit, $f(x)$ a beta function, $p = 2$. Asymptote at $r = 1$ defines the bounding line.](image)

where $\tau^\text{ext}$ is the externally applied shear stress, given by (6), and $\tau^\text{int}$ is an internal stress that develops in the material at the microscopic level due to the heterogeneity of plastic deformation (Wilson & Konnan 1964; Wilson 1965). For determining the internal shear stress, in terms of the present notation, Lin (1977) proposed an expression

$$
\dot{\gamma}^\text{int} = -H^\text{int}\gamma^p,
$$

(43)

where $H^\text{int}$ may depend on the state of hardening.

In terms of $\tau$, it can now be assumed that the glide systems exhibit unbounded flow under quasistatic loading at some value of $\tau = \tau_0$. According to (29), this means that $\bar{f}(x)$ has a finite support $[0, x_0]$, where $x_0 = \tau_0/\bar{s}$. If the reference stress $\bar{s}$ is taken to be $\tau_0$, the support of $\bar{f}(x)$ is precisely $[0, 1]$. To elaborate on this point further, first one can recast (29) as

$$
\dot{\gamma}^p(t) = \dot{r}(t)/H^p(t),
$$

(44)

where

$$
H^p(t) = \{1 - \bar{P}[r(t)]\}/\gamma_p^p f_0[r(t), t].
$$

(45)

But, since $\dot{r}(t) = \dot{r}^\text{ext}(t) + \dot{r}^\text{int}(t)$, with the aid of (43), one can express (44) in terms of $\dot{r}^\text{ext}(t)$ as

$$
\dot{\gamma}^p(t) = \dot{r}^\text{ext}(t)/[H^p(t) + H^\text{int} \equiv \dot{r}^\text{ext}(t)/H(t),
$$

(46)
where $H(t)$ is a tangent plastic modulus. Since $\dot{f}(x)$ has a support $[0, 1]$, as $r(t) \to 1$, $H^p(t) \to 0$, and $H(t) \to H^{\text{int}}$. This in turn implies that the monotonic $r^{\text{exl}}-\gamma^p$ curve for a glide system approaches the bounding line $1 + \tau(\gamma^p)$, where $\tau(\gamma^p)$ denotes the curve resulting from integration of (43). If, for instance, $H^{\text{int}}$ is taken to be constant, the bounding line is a straight line.

![Diagram](image)

**Figure 12.** Normalized uniaxial stress–plastic strain curves showing loading rate effect with the assumption $\dot{f}(x) = \delta(x)$. The inviscid limit occurs as $\dot{r} \to 0$.

A representative family of materials is obtained by having $\dot{f}(x)$ defined by a $\beta$-density-function given as

$$\dot{f}(x) \equiv x^{p-1}(1-x)^{q-1}/B(p, q),$$

(47)

where $B(p, q)$ is the $\beta$-function (figure 10). Some quasistatic uniaxial stress–strain curves for this family are shown in figure 11. These curves were determined by means of a numerical integration of (1). It is evident from this diagram that, by varying the parameters $p$ and $q$, one can obtain a variety of shapes that may be used to fit experimental data for different materials. Figure 12 shows the behaviour of a material at finite loading rates.

An experimental record of an uniaxial cyclic test specifically devised to illustrate the recovery effect is shown in figure 13. This experiment was performed with a mild steel specimen with slowly applied loads. In the strain range above 3%, a series of loops of different widths may be noted. These loops coincide on the left for ease of comparison with each other of their reloading paths such as $B_1A_1$, $B_2A_2$, etc. Here it is to be noted that virtually no plastic deformation is observed during an unloading-reloading path $A_1B_1A_2$ before reaching the envelope. The material
Fig. 13. Uniaxial stress-strain diagram for a mild steel tubular specimen. Experiment designed to exhibit the recovery effect.

'remembers' the unloading point A₃, and a sharp knee is traced at that point upon reloading. On the other hand, if a cyclic loading process is allowed to proceed into the compressive zone, such as points B₂ to B₇ in the figure, the reloading curves show a gradual transition from the elastic into a plastic régime. This transition becomes more and more gradual as the amount of plastic deformation into the compressive zone is increased. Eventually, the reloading curves in the tensile direction tend to retrace each other. This observation leads to the conclusion that a large plastic deformation in one direction of loading wipes out from the memory of the material its previous loading history. In such cases, a reloading curve approaches asymptotically the virgin shape.

The analytic model proposed in this paper exhibits the above type of behaviour. In the inviscid limit, preloading of a glide system up to a moderate stress τ induces an elastic region in the interval [0, τ] owing to the fact that inside it \( f_0 = 0 \) (see figure 4). In the above experiment, this would correspond to the segment A₁B₁ in figure 13. A subsequent unloading process triggers a recovery toward an unperturbed distribution \( \bar{f}(x) \) (see figure 6). From (33) it follows that \( f_0 \) tends to \( \bar{f}(x) \) at an exponential rate as the amount of plastic deformation in the unloading direction is increased. This accounts for the behaviour observed in figure 13, that the reloading curves tend to the virgin shape as the width of the loop is increased. Some numerically obtained curves for a few cycles of progressively increasing intensity resembling those of the experiment are shown in figure 14. These results show a good qualitative agreement between the proposed material model and the experimental record.

As the last example, the proposed model is adopted to describe materials that exhibit yield plateau. According to Cottrell (1953), this well known phenomenon is
Figure 14. Computed normalized uniaxial stress-plastic strain diagram for a material with an assumed $\tilde{f}(x) = 2x$ for $0 \leq x \leq 1$ showing a recovery effect similar to that in figure 13.

Figure 15. Normalized uniaxial stress-plastic strain diagram for a material exhibiting a yield plateau, $\tilde{f}(x)$ a $\beta$-function, $p = 2$, $q = 11$. 
related to the interaction between dislocations and solute atoms. When time is allowed for the necessary diffusion (ageing), these interactions cause the solute atoms to segregate around stationary dislocations. For this reason, each dislocation line gathers around it an ‘atmosphere’ of solute atoms in unyielded or in strained-aged condition. Therefore, if the characteristic time for the loading process applied to the specimen is short compared with the relaxation time for the migration process of the solute atoms (2–3 hours), the atmosphere will be immobile, and the dislocations will be anchored along their entire lengths to fixed positions in the lattice. Therefore, to produce a plastic flow, the resolved shear stress has to exceed the resistance $\tau_y$ of the atmosphere of solute atoms. Under such circumstances, the distribution of resistances faced by the dislocation segments is expressed by a Dirac $\delta$-function

$$f(x, t_0) = \delta(x - x_y),$$

with $x_y = \tau_y / \dot{\gamma}$. Some numerical results illustrating this technique are shown in figure 15; which seems to be in good agreement with the typically observed behaviour.

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